AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume XLIX, Number 4

OCTOBER, 1927

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

CONTENTS

THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is $6.00 (foreign postage 25 cents); single numbers $1.75.

A few complete sets of the JOURNAL remain on sale.

Papers intended for publication in the JOURNAL may be sent to any of the Editors.

Editorial communications may be sent to Dr. A. COHEN at the Johns Hopkins University.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

Oskar Bolza

New Results in Elimination.*

By F. Morley and A. B. Coble.

Elimination is a fundamental process in algebra. One of its most general problems may be stated in homogeneous form as follows:

Given j forms,

$$\alpha_{1y}^{m_1},\ \alpha_{2y}^{m_2},\ \cdots,\ \alpha_{jy}^{m_j} \qquad (m_1 \gtrless m_2 \gtrless m_3 \cdots \gtrless m_j)$$

homogeneous and of the orders m indicated in the variables $y_1, y_2, \cdots, y_j$, to find that polynomial in the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_j$ whose vanishing expresses the necessary and sufficient condition that the forms vanish simultaneously for a value system $x_1, x_2, \cdots, x_j$. This polynomial R is the so-called resultant or eliminant i. e. the result of eliminating $y_1 : y_2 : \cdots : y_j$ from the j equations, $\alpha_{1y}^{m_1} = 0$, $\alpha_{2y}^{m_2} = 0, \cdots, \alpha_{jy}^{m_j} = 0$. As thus defined a numerical factor in R is unessential though in certain cases (e. g. if R is included among the members of a complete system of invariants of the given forms) some convention as to such a factor may be desirable. In geometric language $R = 0$ is the necessary and sufficient condition that the j spreads, $\alpha_{1y}^{m_1} = 0, \cdots, \alpha_{jy}^{m_j} = 0$, of dimension $j - 2$ in S_{j-1} have a common point x. The degree of R in the coefficients of a particular form $\alpha_{ky}^{m_k}$ is known to be $m_1 . m_2 . \cdots . m_j / m_k$. A subsidiary problem is that of exhibiting in rational form the common solution x which exists when $R = 0$.

Other problems such as those of "restricted" elimination appear e. g. to express the condition that three conics with two known common points (say conics without terms in y_1^2, y_2^2) shall have a further common point. This memoir is confined to the general problem formulated above.

The binary case, $j = 2$, of elimination of one non-homogeneous variable, or two homogeneous variables, from two equations has been satisfactorily solved. The dialytic determinant of Sylvester provides the resultant in the convenient form now commonly presented. A much earlier method of Bézout† modified later by Cayley ‡) for two forms of the same order, extended by Rosenhain § to different orders, furnishes the resultant as a more contracted determinant. Conditions for further common solutions have been found.¶

* An investigation pursued under the auspices of the Carnegie Institution of Washington, D. C.

† Paris Memoirs (1764). ‡ *Journal für Mathematik*, Vol. 53 (1857.)

§ *Journal für Mathematik*, Vol. 28 (1844).

¶ Further references are found in the *Encyklopädie*, Vol. 1, pp. 245-250; 260-274; and in Pascal's *Repertorium*, Leipzig (Teubner) (1900), Vol. 1, pp. 86-89; 274-278; 322-323.

Until recently however little progress had been made, in the elimination of more variables, toward the formation of a pure resultant, i. e. one free of extraneous factors which contain the coefficients of one or more of the given forms. It is clear that, as the variables are successively eliminated from selected pairs of equations by the dialytic or other method, a result R' is obtained which is the resultant R multiplied by a factor which depends upon the order in which the variables are eliminated and upon the order in which the equations are selected. As this order changes a variety of results R' appear of which R is the G. C. D. Thus elimination of any number of variables is properly regarded as a rational operation but the explicit form of the end result which we seek here has not been exhibited in the general case.*

Little effective progress has been made by the methods of invariant theory such as that devised by Gordan † for three ternary forms.

The first notable extension of the Cayley-Bézout determinant to higher cases was presented by Professor Morley to the Toronto Congress (1924).‡ This furnished the resultant R for three ternary forms of the same order.

It is the purpose of this paper to extend this result to a wide variety of ternary cases and a more limited number of cases in more than three variables. The orders m for which R is explicitly obtained as a determinant are, in ascending magnitude, the following:

$$
\begin{array}{lll}
j=3: & m_1,\ m_2,\ m_2+r \quad (r=0,\ 1,\ \cdots,\ m_1+1) & ; \\
j=4: & m_1,\ m_1,\ m_1,\ m_1+r \quad (r=0,\ 1,\ 2) & , \\
 & m_1,\ m_1,\ m_1+1,\ m_1+1 & , \\
 & m_1,\ m_1+r,\ m_1+r,\ m_1+r \quad (r=1,2) & , \\
 & m_1,\ m_1+1,\ m_1+1,\ m_1+2 & ; \\
j=5: & 3,\ 3,\ 3,\ 3,\ 3 & , \\
 & 2,\ 2,\ 2,\ 2,\ 2+r \quad (r=0,1) & ; \\
j=6: & 2,\ 2,\ 2,\ 2,\ 2,\ 2 & .
\end{array}
$$

Further cases for which some of the orders m are unity are given in § 2.

This is accomplished by the use of a covariant J_k introduced by Morley for the particular case mentioned. J_k may be defined as the sum of jacobians of polars of x as to the given forms whose order in y is k; or more specifically

* For other rational methods see Salmon's *Higher Algebra*, (1876), pp. 79-83.

† *Mathematische Annalen*, Vol. 50 (1897).

‡ Published in this *Journal*, Vol. 47 (1925), pp. 91-97, under the title: "The Eliminant of a Net of Curves."

$$(1) \qquad J_k = \sum_{k_1, \ldots, k_j} (\alpha_1 \alpha_2 \cdots \alpha_j) \alpha_{1x}^{m_1-k_1-1} \alpha_{1y}^{k_1} \cdots \alpha_{jx}^{m_j-k_j-1} \alpha_{jy}^{k_j}$$
$$(k_1 + \cdots + k_j = k).$$

The importance of this particular sum of jacobians is doubtless due to the recurrence relations (4) which it satisfies.

In addition to J_k we use the *modulus* F defined by the given forms, $\alpha_{1y}^{m_1}, \cdots, \alpha_{jy}^{m_j}$, i. e. the totality of algebraic forms comprised under the formula,

$$(2) \qquad F = \beta_{1y}^{k-m_1} \cdot \alpha_{1y}^{m_1} + \beta_{2y}^{k-m_2} \cdot \alpha_{2y}^{m_2} + \cdots + \beta_{jy}^{k-m_j} \cdot \alpha_{jy}^{m_j}$$

where k is any order and $\beta_{1y}^{k-m_1}, \cdots, \beta_{jy}^{k-m_j}$ are *arbitrary* forms of the orders indicated. If in particular the coefficients β all vanish we note that a form which vanishes identically is contained in the modulus.

The method is based on the following fundamental theorem:

(I) *If the equations,* $\alpha_{1y}^{m_1} = 0, \cdots, \alpha_{jy}^{m_j} = 0$, *have a common solution* x, *then all of the covariants* J_k *for* $k = 0, 1, \cdots, m_1 + \cdots + m_j - j$ *are contained in the modulus* F.

For $k = 0, 1, \cdots, m_1 - 1$ the order of J_k in y is too low to permit of an effective expression (2) whence, for these values of k, $J_k \equiv 0$. Thus in Morley's case, $m_1 = m_2 = m_3$, we have the syzygies, $J_{m_1-1} \equiv 0$, $J_{m_1-2} \equiv 0$, which he used to form the resultant.

The identity in y given by the fundamental theorem furnishes a number of equations which contain, in J_k on the left, products of powers of the unknown common solution x, and on the right the unknown coefficients β of a form of the modulus. Additional equations may be obtained by multiplying $\alpha_{1x}^{m_1} = 0, \cdots, \alpha_{jx}^{m_j} = 0$ by such products of powers of $x_1, \cdots, x_j$ as will produce the order, $\sum m_i - j - k$, of J_k in x. From such equations, linear in the products x and the coefficients β, these quantities may be eliminated in the cases mentioned and the resultant R appears as a determinant.

We observe that the coefficients β in the expression (2) of F are no longer unique if $k \geqq m_1 + m_2$. For

$$\beta_{1y}^{k-m_1} \cdot \alpha_{1y}^{m_1} + \beta_{2y}^{k-m_2} \cdot \alpha_{2y}^{m_2} \equiv$$
$$[\beta_{1y}^{k-m_1} + \gamma_y^{k-m_1-m_2} \cdot \alpha_{2y}^{m_2}] \cdot \alpha_{1y}^{m_1} + [\beta_{2y}^{k-m_2} - \gamma_y^{k-m_1-m_2} \cdot \alpha_{1y}^{m_1}] \cdot \alpha_{1y}^{m_2}.$$

If then we hold k at the upper limit $m_1 + m_2 - 1$ which avoids this ambiguity in the coefficients β, the order in x, $\sum m_i - j - k$, also should not exceed $m_1 + m_2 - 1$. Otherwise the additional equations will not be independent since the one obtained by multiplying $\alpha_{1x}^{m_1} = 0$ by $\alpha_{2x}^{m_2}$ coincides with the one

obtained by multiplying $\alpha_{2x}^{m_2}=0$ by $\alpha_{1x}^{m_1}$. Thus our equations are independent only if $\sum m_i - j - k < m_1 + m_2$ $(k = m_1 + m_2 - 1)$ i. e. if

$$m_3 + m_4 + \cdot\cdot\cdot + m_j < m_1 + m_2 + j - 1,$$

an inequality satisfied only by the cases noted above.

The discriminant of a spread, $\alpha_y^m = 0$, in S_{j-1} is the resultant of the j first derivatives of order $m-1$. Hence we provide a determinant expression for the discriminant of any plane curve (Morley), of any surface, of the cubic and quartic spread in S_4, and of the cubic spread in S_5.

The details of the argument are given in the sections which follow. A proof of the fundamental theorem (I) appears in § 1. In § 2 we show that the equations obtained from $J_{\sum m_i - j - (m_1+m_2-1)}$ and the additional equations obtained from the given forms permit of eliminating the coefficients β and products x to obtain a determinant R. In § 3 it is proved that R has the same degree in the coefficients of the given forms as the resultant and therefore either R is the resultant or $R \equiv 0$. But $R \not\equiv 0$ since it is verified that $R = \pm 1$ for the particular forms, $y_1^{m_1}, y_2^{m_2}, \cdot\cdot\cdot, y_j^{m_j}$. There is given in § 4 an interesting extension of the fundamental theorem which applies to $j+1$ general forms. In § 5 we prove that the resultant R, when properly bordered, furnishes in general the unique apolar form of the j given forms and therefore, when $R=0$, a power of ξ_x, the common solution. The existence of other forms of the resultant, also determinants, here becomes clear. These alternative forms are frequently determinants of lower order than those given in § 2.

The extension of the fundamental theorem which appears in § 4 is in effect a generalized form of the syzygy which Morley used as a starting point. In § 6 a new proof along his earlier line is given and the application to the formation of resultants appears as a corollary.

1. *Proofs of the fundamental theorem.* The first proof of theorem (I) is based only on the determinant identities and on the recursion formulae (4) below. These persist for any number of variables so that it will be sufficient to carry the proof through for four variables and then to state the general result.

We taken then four quaternary forms or surfaces,

$$(1) \qquad \alpha_y^m,\ \beta_y^n,\ \gamma_y^p,\ \delta_y^q \qquad (m \gtrless n \gtrless p \gtrless q),$$

all on a point x. The form J_k, the sum of jacobians of polars of x whose degree in y is k, we write as

$$(2)\qquad J_k \equiv (\alpha\beta\gamma\delta)\left({m-1,\,n-1,\,p-1,\,q-1 \atop k}\right), \qquad (k > 0 \text{ else } J_k = 0).$$

In this $(\alpha\beta\gamma\delta)$ is the usual symbolic determinant and the second parenthesis represents a sum of terms, namely

$$(3)\qquad \left({m-1,\,n-1,\,p-1,\,q-1 \atop k}\right)$$
$$\equiv \sum_{k_1,\ldots,k_4} \alpha_x^{m-k_1-1}\alpha_y^{k_1}\beta_x^{n-k_2-1}\beta_y^{k_2}\,\gamma_x^{p-k_3-1}\gamma_y^{k_3}\delta_x^{q-k_4-1}\delta_y^{k_4}$$
$$(k_1 + k_2 + k_3 + k_4 = k).$$

We observe that every term of this sum contains a factor α_y or, if not α_y, a factor α_x^{m-1}; also that every term contains a factor α_x, or, if not α_x, a factor α_y^{m-1}. Thus

$$(4.1\alpha)\qquad \left({m-1,\,n-1,\,p-1,\,q-1 \atop k}\right) = \alpha_y\left({m-2,\,n-1,\,p-1,\,q-1 \atop k-1}\right) + \alpha_x^{m-1}\left({0,\,n-1,\,p-1,\,q-1 \atop k}\right);$$

$$(4.2\alpha)\qquad \left({m-1,\,n-1,\,p-1,\,q-1 \atop k}\right) = \alpha_x\left({m-2,\,n-1,\,p-1,\,q-1 \atop k}\right) + \alpha_y^{m-1}\left({0,\,n-1,\,p-1,\,q-1 \atop k-m+1}\right);$$

and similar formulae are valid for β, γ, and δ. If any of the integers in these parentheses are negative the corresponding terms do not appear.

Let ξ_x be an arbitrary linear form which represents a plane ξ not on the point x common to the four surfaces. We multiply J_k by ξ_x and replace $(\alpha\beta\gamma\delta)\xi_x$ from

$$(5)\qquad (\alpha\beta\gamma\delta)\xi_x \equiv (\alpha\beta\gamma\xi)\delta_x - (\alpha\beta\delta\xi)\gamma_x + (\alpha\gamma\delta\xi)\beta_x - (\beta\gamma\delta\xi)\alpha_x.$$

If now to the last term in α_x we apply (4.1α) the second term on the right of (4.1α) contains the factor α_x^m which by hypothesis vanishes. Thus

$$(6)\qquad \xi_x \cdot J_k = (\alpha\beta\gamma\xi)\left({m-1,\,n-1,\,p-1,\,q-2 \atop k-1}\right)\delta_x\delta_y - (\alpha\beta\delta\xi)\left({m-1,\,n-1,\,p-2,\,q-1 \atop k-1}\right)\gamma_x\gamma_y$$
$$+ (\alpha\gamma\delta\xi)\left({m-1,\,n-2,\,p-1,\,q-1 \atop k-1}\right)\beta_x\beta_y - (\beta\gamma\delta\xi)\left({m-2,\,n-1,\,p-1,\,q-1 \atop k-1}\right)\alpha_x\alpha_y.$$

In the first term on the right of (6) substitute for $(\alpha\beta\gamma\xi)\delta_y$ from

$$(\alpha\beta\gamma\xi)\delta_y \equiv (\alpha\beta\gamma\delta)\xi_y + (\alpha\beta\delta\xi)\gamma_y - (\alpha\gamma\delta\xi)\beta_y + (\beta\gamma\delta\xi)\alpha_y.$$

We have then on the right one term in $(\alpha\beta\gamma\delta)$ and two terms in each of $(\alpha\beta\delta\xi)$, $(\alpha\gamma\delta\xi)$, $(\beta\gamma\delta\xi)$. The pair in $(\alpha\beta\delta\xi)$ is

$$(\alpha\beta\delta\xi)\gamma_y\left[\delta_x\left({m-1,\,n-1,\,p-1,\,q-2 \atop k-1}\right) - \gamma_x\left({m-1,\,n-1,\,p-2,\,q-1 \atop k-1}\right)\right].$$

To reduce this pair we apply (4.2γ) to the first parenthesis and (4.2δ) to the second and obtain

$$(\alpha\beta\delta\xi)\left[\delta_x\left({m-1,\,n-1,\,0,\,q-2 \atop k-p}\right)\cdot\gamma_y^p - \gamma_y\gamma_x\delta_y^{q-1}\left({m-1,\,n-1,\,p-2,\,0 \atop k-q}\right)\right].$$

With similar modifications in the other two pairs of terms we have

$$
\begin{aligned}
(7)\quad \xi_x \cdot J_k = \xi_y \cdot (\alpha\beta\gamma\delta) &\tbinom{m-1,\, n-1,\, p-1,\, q-2}{k-1}\delta_x \\
&+ (\alpha\beta\delta\xi)\left[\delta_x \tbinom{m-1,\, n-1,\, 0,\, q-2}{k-p} \cdot \gamma_y{}^p - \gamma_y\gamma_x\delta_y{}^{q-1}\tbinom{m-1,\, n-1,\, p-2,\, 0}{k-q}\right] \\
&- (\alpha\gamma\delta\xi)\left[\delta_x \tbinom{m-1,\, 0,\, n-1,\, q-2}{k-n} \cdot \beta_y{}^n - \beta_y\beta_x\delta_y{}^{q-1}\tbinom{m-1,\, n-2,\, p-1,\, 0}{k-q}\right] \\
&+ (\beta\gamma\delta\xi)\left[\delta_x \tbinom{0,\, n-1,\, p-1,\, q-2}{k-m} \cdot \alpha_y{}^m - \alpha_y\alpha_x\delta_y{}^{q-1}\tbinom{m-2,\, n-1,\, p-1,\, 0}{k-q}\right].
\end{aligned}
$$

We observe at this point that if $k < q$ the three terms in $\delta_y{}^{q-1}$ will not appear because of the negative $(k - q)$ in the parentheses. Then in the first term on the right of (7) we can replace, by virtue of (4.28),

$$\tbinom{m-1,\, n-1,\, p-1,\, q-2}{k-1}\delta_x \text{ by } \tbinom{m-1,\, n-1,\, p-1,\, q-1}{k-1}$$

and we have a preliminary expression for J_k in terms of the modulus $(J_{k-1}, \alpha, \beta, \gamma, \delta)$ which reads:

$$
\begin{aligned}
(8)\quad \xi_x \cdot J_k = \xi_y \cdot J_{k-1} &+ (\alpha\beta\delta\xi)\tbinom{m-1,\, n-1,\, 0,\, q-2}{k-p}\delta_x \cdot \gamma_y{}^p \\
&-(\alpha\gamma\delta\xi)\tbinom{m-1,\, 0,\, p-1,\, q-2}{k-n}\delta_x \cdot \beta_y{}^n + (\beta\gamma\delta\xi)\tbinom{0,\, n-1,\, p-1,\, q-2}{k-m}\delta_x \cdot \alpha_y{}^m, \\
&\qquad (k < q).
\end{aligned}
$$

In this if $k < p$ the term in $\gamma_y{}^p$ also will not occur; if $k < n$, only the term in $\alpha_y{}^m$ appears; and if further $k < m$ we shall have merely that

$$(9)\qquad (k < m) \qquad \xi_x \cdot J_k = \xi_y \cdot J_{k-1}.$$

Since J_0, the ordinary jacobian, vanishes at the common point x the recursion formula (9) yields, by repeated multiplication by ξ_x, the result

$$(10)\qquad (k < m) \qquad J_k \equiv 0.$$

Thus for three ternary forms and equal orders $m = n = p$ we have Morley's syzygies:

$$J_{m-1} \equiv 0, \quad J_{m-2} \equiv 0.$$

If $k = m < n$, formula (8) becomes

$$(11)\qquad \xi_x \cdot J_m \equiv (\beta\gamma\delta\xi)\tbinom{0,\, n-1,\, p-1,\, q-1}{0} \cdot \alpha_y{}^m \qquad (m < n \gtrless p \gtrless q),$$

a rather remarkable expression for a general surface $\alpha_y{}^m$ in terms of a point x on it and any three other surfaces of higher order which also are on x.

We return to the general formula (7) and modify the first term on the right by substituting for $\tbinom{m-1,\, n-1,\, p-1,\, q-2}{k-1}\delta_x$ from (4.28) which brings in a new term in $\delta_y{}^{q-1}$ i. e.

$$\delta_x \binom{m-1,\, n-1,\, p-1,\, q-2}{k-1} = \binom{m-1,\, n-1,\, p-1,\, q-1}{k-1} - \delta_y^{q-1} \binom{m-1,\, n-1,\, p-1,\, 0}{k-q}.$$

We modify also the three terms in (7) which already contain δ_y^{q-1} by using

$$\gamma_x \binom{m-1,\, n-1,\, p-2,\, 0}{k-q} = \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} - \gamma_y^{p-1} \binom{m-1,\, n-1,\, 0,\, 0}{k-p-q+1},$$

$$\beta_x \binom{m-1,\, n-2,\, p-1,\, 0}{k-q} = \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} - \beta_y^{n-1} \binom{m-1,\, 0,\, p-1,\, 0}{k-n-q+1},$$

$$\alpha_x \binom{m-2,\, n-1,\, p-1,\, 0}{k-q} = \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} - \alpha_y^{m-1} \binom{0,\, n-1,\, p-1,\, 0}{k-m-q+1}.$$

Then the right member of (7) will contain first a term $\xi_y \cdot J_{k-1}$; second, four terms in $\delta_y^{q-1} \binom{m-1,\, n-1,\, p-1,\, 0}{k-q}$ with coefficients,

$$-(\alpha\beta\gamma\delta)\xi_y - (\alpha\beta\delta\xi)\gamma_y + (\alpha\gamma\delta\xi)\beta_y - (\beta\gamma\delta\xi)\alpha_y = -(\alpha\beta\gamma\xi)\delta_y,$$

which therefore contribute the single term

$$-(\alpha\beta\gamma\xi) \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} \cdot \delta_y^q;$$

third, two terms in $(\alpha\beta\delta\xi) \cdot \gamma_y^p$ with coefficients

$$\delta_x \binom{m-1,\, n-1,\, 0,\, q-2}{k-p} + \delta_y^{q-1} \binom{m-1,\, n-1,\, 0,\, 0}{k-p-q+1} = \binom{m-1,\, n-1,\, 0,\, q-1}{k-p}$$

which therefore contribute the single term $(\alpha\beta\delta\xi) \binom{m-1,\, n-1,\, 0,\, q-1}{k-p} \cdot \gamma_y^p$; and finally two similar terms in each of $(\alpha\gamma\delta\xi) \cdot \beta_y^n$ and $(\beta\gamma\delta\xi) \cdot \alpha_y^m$. Thus (7) takes the recursion form:

$$(12) \quad \begin{aligned} \xi_x \cdot J_k \equiv\ & \xi_y \cdot J_{k-1} \\ & - (\alpha\beta\gamma\xi) \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} \cdot \delta_y^q + (\alpha\beta\delta\xi) \binom{m-1,\, n-1,\, 0,\, q-1}{k-p} \cdot \gamma_y^p \\ & - (\alpha\gamma\delta\xi) \binom{m-1,\, 0,\, p-1,\, q-1}{k-n} \cdot \beta_y^n + (\beta\gamma\delta\xi) \binom{0,\, n-1,\, p-1,\, q-1}{k-m} \cdot \alpha_y^m. \end{aligned}$$

The formula (12) furnishes an expression for J_k in terms of the modulus $(J_{k-1}, \alpha_y^m, \beta_y^n, \gamma_y^p, \delta_y^q)$. If we multiply (12) by ξ_x and replace on the right $\xi_x \cdot J_{k-1}$ by using (12) for the value $k-1$ we have an expression for J_k in terms of the modulus $(J_{k-2}, \alpha_y^m, \beta_y^n, \gamma_y^p, \delta_y^q)$. On continuing this process and recalling from (10) that $J_k \equiv 0$ if $k < m$ we find the following expression for J_k in terms of the smaller modulus $(\alpha_y^m, \beta_y^n, \gamma_y^p, \delta_y^q)$:

$$(13) \quad \begin{aligned} \xi_x^{k-m+1} \cdot J_k \equiv\ & \\ & - (\alpha\beta\gamma\xi) \left\{ \sum_{r=0}^{k-m} \binom{m-1,\, n-1,\, p-1,\, 0}{k-q-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \delta_y^q \\ & + (\alpha\beta\delta\xi) \left\{ \sum_{r=0}^{k-m} \binom{m-1,\, n-1,\, 0,\, q-1}{k-p-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \gamma_y^p \\ & - (\alpha\gamma\delta\xi) \left\{ \sum_{r=0}^{k-m} \binom{m-1,\, 0,\, p-1,\, q-1}{k-n-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \beta_y^n \\ & + (\beta\gamma\delta\xi) \left\{ \sum_{r=0}^{k-m} \binom{0,\, n-1,\, p-1,\, q-1}{k-m-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \alpha_y^m. \end{aligned}$$

If now we adopt for J_k the more specific notation

$$(14) \qquad J_k \equiv J_k\,(\alpha^m,\ \beta^n\ \gamma^p,\ \delta^q)$$

then it is clear from the definition of J_k that the coefficients of the modulus in (13) are also of the J-type and we write the final form of the identity:

$$(15) \qquad \xi_x^{k-m+1}\cdot J_k(\alpha^m,\ \beta^n,\ \gamma^p,\ \delta^q) \equiv -J_{k-q}(^m,\ \beta^n,\ \gamma^p,\ \xi^{k-m+1})\cdot\delta_y{}^q$$
$$+ J_{k-p}(\alpha^m,\ \beta^n,\ \delta^q,\ \xi^{k-m+1})\cdot\gamma_y{}^p - J_{k-n}(\alpha^m,\ \gamma^p,\ \delta^q,\ \xi^{k-m+1})\cdot\beta_y{}^n$$
$$+ J_{k-m}(\beta^n,\ \gamma^p,\ \delta^q,\ \xi^{k-m+1})\cdot\alpha_y{}^m.$$

We extend this at once to the general case and give the following more explicit statement of the theorem (I):

If the spreads $\alpha_{1y}{}^{m_1}$, $\alpha_{2y}{}^{m_2}$, $\cdots$, $\alpha_{jy}{}^{m_j}$ in S_{j-1} have a common point x, the form $J_k(\alpha_1{}^{m_1}, \cdots, \alpha_j{}^{m_j})$ of order k in y belongs to the modulus defined by the j given forms. If ξ_y is a fixed linear form such that $\xi_x \neq 0$, the explicit expression for J_k in terms of the modulus is

$$(16) \qquad \xi_x^{k-m_1+1}\cdot J_k(\alpha_1{}^{m_1},\cdots,\alpha_j{}^{m_j}) \equiv$$
$$\sum_{s=1}^{j}(-1)^{j-s+1}J_{k-m_s}\,(\alpha_1{}^{m_1},\ \cdots,\ \alpha_{s-1}{}^{m_{s-1}},\ \alpha_{s+1}{}^{m_{s+1}},\ \cdots,\ \alpha_j{}^{m_j},\ \xi^{k-m_1+1})\cdot\alpha_{sy}{}^{m_s}.$$

2. *Formation of the Resultant.* If the j equations, $\alpha_{iy}{}^{m_i}=0$ $(i=1, \cdots, j)$ are given subject merely to the condition that they have an unknown common solution x, then, in the syzygy [i. e. the identical relation of order k in y (16)] where ξ is fixed, we have the constant $\xi_x^{k-m_1+1}$ and the forms J_{k-m_s} of order $k-m_s$ in y whose coefficients also contain the unknown x. Thus after division by $\xi_x^{k-m_1+1}$ we have an identity of order k in y of the form

$$(17) \qquad J_k(\alpha_1{}^{m_1},\cdots,\alpha_j{}^{m_j}) + \beta_{1y}{}^{k-m_1}\cdot\alpha_{1y}{}^{m_1} + \cdots + \beta_{jy}{}^{k-m_j}\cdot\alpha_{jy}{}^{m_j} \equiv 0$$

in which there occur the unknown coefficients of the forms β and (in J_k) the combinations, also unknown, of the coordinates x, which are of degree $\sum m_i - j - k$. We seek to eliminate these t unknowns from a set of t equations linear in them. For this it is essential that the values of the unknowns be unique and that the additional equations which may be obtained from $\alpha_{1x}{}^{m_1}=0$, $\alpha_{2x}{}^{m_2}=0$, $\cdots$ by multiplication with such combinations of x as raise them to the degree $\sum m_i - j - k$ shall themselves be linearly independent. As explained in the introduction this leads to the inequalities

$$\sum m_i - j - k \leqq m_1 + m_2 - 1, \quad k \leqq m_1 + m_2; \text{ or to}$$
$$m_3 + \cdots + m_j < j + k, \quad k < m_1 + m_2.$$

Thus the most favorable range for the orders $m_3, \cdots, m_j$ is obtained by taking the maximum value for k, i. e. $m_1 + m_2 - 1$; and the original inequalities and the one just deduced are

$$(18) \qquad m_1 \leqq m_2 \leqq \cdots \leqq m_j,$$
$$m_3 + \cdots + m_j < m_1 + m_2 + (j-1).$$

The case $m_i = 1$ may be discarded since if any of the given forms are linear the elimination may at once be reduced to a lower dimension. To this point however we shall recur. Since $m_3 + m_4 \geqq m_1 + m_2$ the largest range for $m_5, \cdots, m_j$ occurs when $m_1 = m_2 = m_3 = m_4$ when

$$m_5 + \cdots + m_j < j - 1.$$

Since $m_i \geqq 2$ the largest range for j appears when $m_i = 2$ and therefore when $2(j-4) < j - 1$ or $j < 7$. For $j = 6$ and the most favorable cases indicated we have $m_5 + m_6 < 5$ or $m_5 = m_6 = 2$ and necessarily $m_1 = \cdots = m_4 = 2$. For $j = 5$ and $m_3 + m_4 + m_5 < m_1 + m_2 + 4$ the largest value of m_5 is obtained when $m_3 + m_4 = m_1 + m_2$ i. e. when $m_1 = \cdots = m_4$ and $m_5 < 4$. If $m_5 = 3$, $m_1, \cdots, m_4$ must all be 3 or 2; if $m_5 = 2$, $m_1, \cdots, m_4$ must all be 2. For $j = 4$ and $m_3 + m_4 < m_1 + m_2 + 3$ it is convenient to set $m_2 = m_1 + r$, $m_3 = m_1 + r + s$, $m_4 = m_1 + r + s + t$ $(r, s, t \geqq 0)$. Then the inequality becomes $r + 2s + t < 3$ whose seven solutions, $(r, s, t) =$ $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, $(2, 0, 0)$, $(0, 0, 2)$, $(1, 0, 1)$, $(0, 1, 0)$ furnish the cases mentioned in the introduction. For $j = 3$ and $m_3 < m_1 + m_2 + 2$ we set $m_3 = m_2 + r$ and find that $r < m_1 + 2$. We have thus accounted for the range of cases listed in the introduction.

If one of the forms and therefore necessarily the first is linear the elimination may in a sense be reduced to elimination in a lower dimension. The procedure for space for forms α_y, β_y^n, γ_y^p, δ_y^q would be as follows. If z, z', z'' are the vertices of a fixed triangle on the given plane α, any point y of the plane is given by $y = \lambda z + \lambda' z' + \lambda'' z''$. On substituting this value of y in the other three forms we have three ternary forms of orders n, p, q in $\lambda, \lambda', \lambda''$ whose eliminant for λ is the required resultant of the four original forms. This resultant however is of degree npq in each of z, z', z'' and in order to express it in terms of the original coefficients α we should have to modify it in such a way as to exhibit the z, z', z'' only in the combinations $\alpha_i = | \, z z' z'' \, |_i$.

Such modification would not in general be practicable and thus it is desirable to see how many cases in which at least one form is linear come under our general method. If then we set $m_1 = 1$, $m_2 = 1 + r_1$, $m_3 = 1 + r_1 + r_2, \cdots$ $(r_1, \cdots, r_{j-1} \geqq 0)$ the original inequality becomes

$$(j-3)r_1 + (j-2)r_2 + (j-3)r_3 + j-4)r_4 \\ + \cdots + 3r_{j-3} + 2r_{j-2} + r_{j-1} < 3.$$

A discussion of this inequality shows that for $j \geqq 6$ there are always four cases for which $(m_{j-1}, m_j) = (1, 1), (1, 2), (1, 3), (2, 2)$ and the other m's all are unity. If in the first three of these the last form is $\alpha_{jy}{}^r$ $(r = 1, 2, 3)$, the resultant obviously is $(\alpha_1\alpha_2 \cdots \alpha_{j-1}\alpha_j)^r$. In the last of the four cases we form the resultant of two binary quadratics in symbolic form and then immediately write it in the symbolic form for the higher case by using the Clebsch translation principle. For $j = 3, 4$ we find precisely the cases already listed in the introduction for which $m_1 = 1$. For $j = 5$ there is in addition to the four cases which always occur when $j \geqq 6$ just one case 1, 2, 2, 2, 2. Our method will in this case furnish a resultant explicit in the coefficients α_1 of the plane α_{1y}, or if these coefficients be replaced by variable coordinates ξ it will furnish the equation of the 16 common points of four quadrics in space S_4.

We consider now the cases listed in the introduction and begin with the three ternary forms of orders

$$j = 3: \qquad m_1,\ m_2,\ m_2 + r \qquad (r = 0, 1, \cdots, m_1 + 1).$$

The fundamental syzygy now reads:

$$J_k + \beta_{1y}{}^{k-m_1} \cdot \alpha_{1y}{}^{m_1} + \beta_{2y}{}^{k-m_2} \cdot \alpha_{2y}{}^{m_2} + \beta_{3y}{}^{k-m_2-r} \cdot \alpha_{3y}{}^{m_2+r} \equiv 0.$$

It is convenient to obtain as many independent equations as possible from $\alpha_{1x}{}^{m_1} = 0$, $\alpha_{2x}{}^{m_2} = 0$, $\alpha_{3x}{}^{m_2+r} = 0$ by multiplication with products of powers of x since such equations contain the coefficients $\alpha_1, \alpha_2, \alpha_3$ in the simplest way and therefore we allow x to enter to the maximum order $m_1 + m_2 - 1$. Since the jacobian of the three forms has the order $m_1 + 2m_2 + r - 3$ the order of J_k in y is $k = m_2 + r - 2$. Thus the syzygy (an identity in y) yields $\binom{k+2}{2}$ equations linear in the $\binom{m_1+m_2+1}{2}$ products π_x of powers of x of order $m_1 + m_2 - 1$ and linear in the coefficients $\beta_1, \beta_2, \beta_3$. But for particular values of m_2 and r some or all of the forms β will vanish identically so that the linear equations will differ formally in certain cases. Thus since $k = m_2 + r - 2$ then $k - m_2 - r < 0$ and $\beta_{3y}{}^{k-m_2-r}$ is always identically zero.

If $r < 2$, $k - m_2 < 0$ and $\beta_{2y}{}^{k-m_2} \equiv 0$. If $r < 2$ and $k = m_2 + r - 2 < m_1$ where also $m_1 \gtrless m_2$ then $\beta_{1y}{}^{k-m_1} \equiv 0$. For $r = 0$ this yields two cases m_1, m_1, m_1 and $m_1, m_1 + 1, m_1 + 1$ the first of which is treated by Morley and will not be considered further here. For $r = 1$ we have one case $m_1, m_1, m_1 + 1$. The last two cases we combine under

Case III$_a$: $\qquad m_{,1}\, m_1 + r, m_1 + 1 \qquad (r = 0, 1) \;:\; J_{m_1-1} \equiv 0.$

If $r < 2$ and $k = m_2 + r - 2 \geqq m_1$ then always $m_2 \gtreqless m_1$. Thus m_2 may have any order and r may be 0, 1 which yields

Case III$_b$: $\qquad m_1, m_2, m_2 + r \quad (r = 0, 1) \;:\; J_{m_2-2+r} \equiv \beta_{1y}{}^{m_2-m_1-2+r} \cdot \alpha_{1y}{}^{m_1}$

$$(m_2 \geqq m_1 + 2 - r),$$

where the last inequality bars Case III$_a$.

For $2 < r < m_1 + 2$ we have the more general

Case III$_c$: $\qquad m_1, m_2, m_2 + r \;(r = 2, \cdots, m_1 + 1) \;:\; J_{m_2-2+r}$

$$\equiv \beta_{1y}{}^{m_2-m_1-2+r} \cdot \alpha_{1y}{}^{m_1} + \beta_{2y}{}^{r-2} \cdot \alpha_{2y}{}^{m_2}.$$

The linear equations obtained from the syzygy, and the additional equations linear in π_x alone obtained by raising the orders of $\alpha_{1x}{}^{m_1} = 0$, $\alpha_{2x}{}^{m_2} = 0$, $\alpha_{3x}{}^{m_3} = 0$ to $m_1 + m_2 - 1$ have the matrices indicated below.

Case III$_a$: $(r = 0, 1)$ $\qquad$ Case III$_b$: $(r = 0, 1)$

$$(19) \quad \begin{array}{c|c|} & \binom{2m_1+r+1}{2} \\ \binom{m_1+1}{2} & |\alpha_1\alpha_2\alpha_3| \\ \binom{m_1+r+1}{2} & |\alpha_1| \\ \binom{m_1+1}{2} & |\alpha_2| \\ \binom{m_1+r}{2} & |\alpha_3| \end{array} \;; \quad \begin{array}{c|cc|} & \binom{m_1+m_2+1}{2} & \binom{m_2-m_1+r}{2} \\ \binom{m_2+r}{2} & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| \\ \binom{m_2+1}{2} & |\alpha_1| & |0| \\ \binom{m_1+1}{2} & |\alpha_2| & |0| \\ \binom{m_1+1-r}{2} & |\alpha_3| & |0| \end{array} \;;$$

Case III$_c$:

$$\begin{array}{c|ccc|} & \binom{m_1+m_2+1}{2} & \binom{m_2+r-m_1}{2} & \binom{r}{2} \\ \binom{m_2+r}{2} & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| \\ \binom{m_2+1}{2} & |\alpha_1| & |0| & |0| \\ \binom{m_1+1}{2} & |\alpha_1| & |0| & |0| \\ \binom{m_1+1-r}{2} & |\alpha_3| & |0| & |0| \end{array} \;.$$

The numbers at the top of each matrix give the number of products π_x, of coefficients β_1, and of coefficients β_2 eliminated i. e. the number of columns

of the interior matrices. The numbers at the left indicate the number of equations obtained from the syzygy and from each of the given equations in order. The interior matrices have elements which are linear in the coefficients indicated except for the matrices $|\,0\,|$ whose elements are zero.

We have to show first that these three matrices are square and thus are determinants which we identify later with the resultant. It is easy to verify that

$$\binom{2m_1+r+1}{2}+\binom{r}{2}=2\binom{m_1+1}{2}+\binom{m_1+r}{2}+\binom{m_1+r+1}{2},$$

$$(20)\qquad \binom{m_1+m_2+1}{2}+\binom{m_2-m_1+r}{2}+\binom{r}{2}=\binom{m_2+r}{2}+\binom{m_2+1}{2}+\binom{m_1+1}{2}+\binom{m_1+1-r}{2}.$$

For $\binom{r}{2}=0$ i. e. $r=0,1$ these prove at once that the matrices III_a and III_b are square, and the second one proves the same for III_c for any value of r.

To the four cases for $j=4$ listed in the introduction we attach the syzygies:

IV_a: $J_{2m_1+r-3}+\beta_{1y}^{m_1+r-3}\cdot\alpha_{1y}^{m_1}+\cdots+\beta_{3y}^{m_1+r-3}\cdot\alpha_{3y}^{m_1}+\beta_{4y}^{m_1-3}\cdot\alpha_{4y}^{m_1+r}\equiv 0;$

IV_b: $J_{2m_1-1}+\beta_{1y}^{m_1-1}\cdot\alpha_{1y}^{m_1}+\beta_{2y}^{m_1-1}\cdot\alpha_{2y}^{m_1}+\beta_{3y}^{m_1-2}\cdot\alpha_{3y}^{m_1+1}+\beta_{4y}^{m_1-2}\cdot\alpha_{4y}^{m_1+1}\equiv 0$

IV_c: $J_{2m_1+2r-3}+\beta_{1y}^{m_1+2r-3}\cdot\alpha_{1y}^{m_1}+\beta_{2y}^{m_1+r-3}\cdot\alpha_{2y}^{m_1+r}+\cdots\beta_{4y}^{m_1+r-3}\cdot\alpha_{4y}^{m_1+r}\equiv 0$

IV_d: $J_{2m_1}+\beta_{1y}^{m_1}\cdot\alpha_{1y}^{m_1}+\beta_{2y}^{m_1-1}\cdot\alpha_{2y}^{m_1+1}+\beta_{3y}^{m_1-1}\cdot\alpha_{3y}^{m_1+1}+\beta_{4y}^{m_1-2}\cdot\alpha_{4y}^{m_1+2}\equiv 0$

The matrices of the corresponding systems of linear equations are:

IV_a $(r=0,1,2)$:

$$(21)\qquad \begin{array}{c|ccccc|}
 & \binom{2m_1+2}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} & \binom{m_1}{3} \\
\binom{2m_1+r}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| \\
\binom{m_1+2}{3} & |\alpha_1| & |0| & |0| & |0| & |0| \\
\binom{m_1+2}{3} & |\alpha_2| & |0| & |0| & |0| & |0| \\
\binom{m_1+2}{3} & |\alpha_3| & |0| & |0| & |0| & |0| \\
\binom{m_1-r+2}{3} & |\alpha_4| & |0| & |0| & |0| & |0|
\end{array};$$

IV_b:

$$\begin{array}{c|ccccc|}
 & \binom{2m_1+2}{3} & \binom{m_1+2}{3} & \binom{m_1+2}{3} & \binom{m_1+1}{3} & \binom{m_1+1}{3} \\
\binom{2m_1+2}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| \\
\binom{m_1+2}{3} & |\alpha_1| & & & & \\
\binom{m_1+2}{3} & |\alpha_2| & & 0 & & \\
\binom{m_1+1}{3} & |\alpha_2| & & & & \\
\binom{m_1+1}{3} & |\alpha_4| & & & &
\end{array};$$

IV_c $(r=1, 2)$:

$$\begin{array}{c|ccccc|}
 & \binom{2m_1+r+2}{3} & \binom{m_1+2r}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} \\
\binom{2m_1+2r}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| \\
\binom{m_1+r+2}{3} & |\alpha_1| & & & & \\
\binom{m_1+2}{3} & |\alpha_2| & & 0 & & \\
\binom{m_1+2}{3} & |\alpha_3| & & & & \\
\binom{m_1+2}{3} & |\alpha_4| & & & &
\end{array};$$

IV_d:

$$\begin{array}{c|ccccc|}
 & \binom{2m_1+3}{3} & \binom{m_1+3}{3} & \binom{m_1+2}{3} & \binom{m_1+2}{3} & \binom{m_1+1}{3} \\
\binom{2m_1+3}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| \\
\binom{m_1+3}{3} & |\alpha_1| & & & & \\
\binom{m_1+2}{3} & |\alpha_2| & & 0 & & \\
\binom{m_1+2}{3} & |\alpha_3| & & & & \\
\binom{m_1+1}{3} & |\alpha_4| & & & &
\end{array}.$$

That here the number of columns is equal to the number of rows, obviously true in the cases IV_b, IV_d, may easily be verified in the cases IV_a, IV_c if in these two cases $\binom{r}{} = 0$, i. e. if $r = 0, 1, 2$.

The determinant IV_a takes a different form when $m_1 = 2$, a form which varies according as $r = 0, 1, 2$. Thus we have three special sub-cases:

IV_{a1} $(m_1 = 2, r = 0)$

(22)
$$\begin{array}{c|c|}
 & 20 \\
4 & |\alpha_1\alpha_2\alpha_3\alpha_4| \\
4 & |\alpha_1| \\
4 & |\alpha_2| \\
4 & |\alpha_3| \\
4 & |\alpha_4|
\end{array};$$

IV_{a2} $(m_1 = 2, r = 1)$

$$\begin{array}{c|cccc|}
 & 20 & 1 & 1 & 1 \\
10 & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\
4 & |\alpha_1| & & & \\
4 & |\alpha_2| & & 0 & \\
4 & |\alpha_3| & & & \\
1 & |\alpha_4| & & &
\end{array};$$

IV_{a3} $(m_1 = 2, r = 2)_3$

$$\begin{array}{c|cccc|}
 & 20 & 4 & 4 & 4 \\
20 & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\
4 & |\alpha_1| & & & \\
4 & |\alpha_2| & & 0 & \\
4 & |\alpha_3| & & &
\end{array}.$$

The further cases listed have numerical orders. The two for $j = 5$, V_a and V_b, have syzygies,

$$J_5 + \beta_{1y}^2 \cdot \alpha_{1y}^3 + \cdots + \beta_{5y}^2 \cdot \alpha_{5y}^3 \equiv 0,$$
$$J_{2+r} + \beta_{1y}^r \cdot \alpha_{1y}^2 + \cdots + \beta_{4y}^r \cdot \alpha_{4y}^2 + \beta_5 \cdot \alpha_{5y}^{2+r} \equiv 0 \quad (r = 0, 1),$$

and corresponding determinants

V_a:

$$(23) \qquad \begin{array}{c|c|ccccc|}
 & 126 & 15 & 15 & 15 & 15 & 15 \\
126 & |\alpha_1 \cdots \alpha_5| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| & |\alpha_5| \\
15 & |\alpha_1| & & & & & \\
15 & |\alpha_2| & & & & & \\
15 & |\alpha_3| & & & 0 & & \\
15 & |\alpha_4| & & & & & \\
15 & |\alpha_5| & & & & &
\end{array} \; ;$$

V_b $(r = 0, 1)$:

$$\begin{array}{c|c|ccccc|}
 & \binom{7}{4} & \binom{r+4}{4} & \binom{r+4}{4} & \binom{r+4}{4} & \binom{r+4}{4} & \binom{4}{4} \\
\binom{r+6}{4} & |\alpha_1 \cdots \alpha_5| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| & |\alpha_5| \\
\binom{5}{4} & |\alpha_1| & & & & & \\
\binom{5}{4} & |\alpha_2| & & & & & \\
\binom{5}{4} & |\alpha_3| & & & 0 & & \\
\binom{5}{4} & |\alpha_4| & & & & & \\
\binom{5-r}{4} & |\alpha_5| & & & & &
\end{array} \; .$$

In V_b for both cases $r = 0, 1$ the matrix is square.

The isolated case 1, 2, 2, 2, 2 for $j = 5$ yields a case V_c for which the syzygy is

$$J_2 + \beta_{1y} \cdot \alpha_{1y} + \beta_2 \cdot \alpha_{2y}^2 + \cdots + \beta_5 \alpha_{5y}^2 \equiv 0,$$

and the determinant is

$$(24) \qquad \begin{array}{c|c|cccc|}
 & 15 & 5 & 1 & \cdots & 1 \\
15 & |\alpha_1 \cdots \alpha_5| & |\alpha_1| & |\alpha_2| & & |\alpha_5| \\
5 & |\alpha_1| & & & & \\
1 & |\alpha_2| & & & 0 & \\
\vdots & & & & & \\
1 & |\alpha_5| & & & &
\end{array} \; .$$

Finally for $j = 6$ we have the one case VI for which the syzygy is

$$J_3 + \beta_{1y} \cdot \alpha_{1y}^2 + \cdots + \beta_{6y} \cdot \alpha_{6y}^2 \equiv 0,$$

and the determinant is

VI:

$$
(25)\qquad
\begin{array}{c}
 \\ 56 \\ 6 \\ 6 \\ \vdots \\ 6
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 56 & 6 & 6 & \cdots & 6 \end{array} \\
\left\|\begin{array}{ccccc}
|\alpha_1\cdots\alpha_6| & |\alpha_1| & |\alpha_2| & \cdots & |\alpha_6| \\
|\alpha_1| & & & & \\
|\alpha_2| & & 0 & & \\
\vdots & & & & \\
|\alpha_6| & & & &
\end{array}\right\|
\end{array}.
$$

In the next section we prove that the determinants (19), (21), (22), (23), (24), and (25) are the resultants of their respective forms.

3. *Identification of the Resultant.* If the j given equations have a common solution the determinants of the preceding section, which obviously have an invariantive character, must vanish since the linear equations from which they were derived are then consistent. If the determinants are not identically zero, they are invariants of the given forms which vanish when the resultant R vanishes and which therefore contain R as a factor. If the degrees of the determinants in the coefficients of the forms coincide with the degrees of R then the determinants coincide with R. We prove first that the determinants have the same degree as R and second that the determinants have, in a particular case, the value ± 1 and therefore do not vanish identically. The argument on each of these two points is given in just a few illustrative examples and the verifications made in the remaining cases are not reproduced here.

All of the determinants given have the typical form

$$
(26)\qquad
\begin{array}{c}
 \\ q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_j
\end{array}
\begin{array}{c}
\begin{array}{ccccc} p_0 & p_1 & p_2 & \cdots & p_j \end{array} \\
\left\|\begin{array}{ccccc}
|\alpha_1\cdots\alpha_j| & |\alpha_1| & |\alpha_2| & \cdots & |\alpha_j| \\
|\alpha_1| & & & & \\
|\alpha_2| & & 0 & & \\
\vdots & & & & \\
|\alpha_j| & & & &
\end{array}\right\|
\end{array}
$$

where the p's and q's represent respectively the number of columns and the number of rows in the interior matrices under or opposite them. Then $\sum_{i=0}^{j} p_i = \sum_{i=0}^{j} q_j = N$. The upper left matrix has elements linear in the coefficients of each of the forms, the other non-zero matrices have for elements precisely the coefficients of the one form indicated. For particular determinants some of these latter matrices do not appear and the corresponding numbers p or q are zero.

In the terms of the expansion of such a determinant there are

$$q_0 - \sum_{i=1}^{j} p_i = p_0 - \sum_{i=1}^{j} q_j = p_0 + q_0 - N$$

elements from the upper left matrix, p_i elements from the matrix $|\alpha_i|$ to the right, and q_i elements from the matrix $|\alpha_i|$ underneath. Hence the degree of the determinant in the coefficients of $\alpha_{i_y}{}^{m_i}$ is

$$p_0 + q_0 + p_i + q_i - N =$$

$$p_0 - q_1 - q_2 \cdots - q_{i-1} + p_i - q_{i+1} \cdots - q_j$$
$$= q_0 - p_1 - p_2 \cdots - p_{i-1} + q_i - p_{i+1} \cdots - p_j.$$

We use the last expression for the degree in α_i since some of the p's are frequently zero.

Thus the determinant of Case III_c in (19) has the following degrees in α_1, α_2, α_3 respectively:

$$\binom{m_2+r}{2} + \binom{m_2+1}{2} - \binom{r}{2} = m_2 \cdot (m_2 + r),$$
$$\binom{m_2+r}{2} + \binom{m_1+1}{2} - \binom{m_2+r-m_1}{2} = m_1 \cdot (m_1 + r),$$
$$\binom{m_2+r}{2} + \binom{m_1+1-r}{2} - \binom{m_2+r-m_1}{2} - \binom{r}{2} = m_1 \cdot m_2,$$

which are the degrees of the resultant.

Similarly in Case IV_c of (21) we find for the degree in α_1 and the equal degrees in α_2, α_3, α_4 that

$$\binom{2m_1+2r}{3} + \binom{m_1+r+2}{3} - 3\binom{m_1+r}{3} = (m_1 + r) \cdot (m_1 + r) \cdot (m_1 + r),$$
$$\binom{2m_1+2r}{3} + \binom{m_1+2}{3} - \binom{m_1+2r}{3} - 2\binom{m_1+r}{3} = m_1 \cdot (m_1 + r) \cdot (m_1 + r) - 12\binom{r}{3},$$

where $\binom{r}{3}$ vanishes for the values of r $(1, 2)$ in this case.

The similar verification which has been made for the other cases is not reproduced here.

We prove that the determinants do not vanish identically by showing that they take the value ± 1 for the particular case when the forms are perfect powers such as

$$y_1^{m_1},\ y_2^{m_2},\ y_3^{m_3},\ \cdots,\ y_j^{m_j}.$$

We use the obvious fact that a determinant whose elements are all zero except for a single element 1 in each row and column has the value ± 1.

Again the verification which has been carried out for all cases, is given only for the cases, III_c in (19) and IV_c in (21), examined above.

For Case III_c the syzygy now reads

$$J_{m_2-2+r} + \beta_{1y}{}^{m_2-m_1-2+r} \cdot y_1{}^{m_1} + \beta_{2y}{}^{r-2} \cdot y_2{}^{m_2} \equiv 0 \quad (r = 2, \cdots, m_1 + 1).$$

In this the forms β are arbitrary forms of their order and the form J_{m_2-2+r} has the simple expression

$$\sum_{k_1,k_2,k_3} y_1{}^{k_1} y_2{}^{k_2} y_3{}^{k_3} x_1{}^{m_1-k_1-1}\, x_2{}^{m_2-k_2-1} x_3{}^{m_2+r-k_3-1}$$

with the following restrictions on the exponents:

$$k_1 + k_2 + k_3 = m_2 - 2 + r,$$

$$\text{(a)} \qquad 0 \leqq k_1 < m_1, \quad 0 \leqq k_2 < m_2, \quad 0 \leqq k_3 < m_2 + r.$$

The product terms of the syzygy contribute terms in $y_1{}^{k_1}y_2{}^{k_2}y_3{}^{k_3}$ for which in the first product $k_1 \geqq m_1$, and in the second $k_2 \geqq m_2$. When we equate to zero the terms of the syzygy in $y_1{}^{k_1}y_2{}^{k_2}y_3{}^{k_3}$, then if $k_1 \geqq m_1$ the term according to (a) does not appear in J. It must appear in the product $\beta_{1y}{}^{m_2-m_1-2+r} \cdot y_1{}^{m_1}$ with a coefficient β_1 which is eliminated leaving an element 1 for the determinant. It also cannot occur in the second product since if $k_1 \geqq m_1$ and $k_2 \geqq m_2$ then $k_1 + k_2 \geqq m_1 + m_2$ which contradicts $k_1 + k_2 + k_3 = m_2 - 2 + r$ $(r = 2, \cdot\cdot\cdot, m_1 + 1)$. Similarly a term for which $k_2 \geqq m_2$ furnishes an equation which contributes a single element $+1$ to the determinant. If finally $k_1 < m_1$ and $k_2 < m_2$ the term can appear only in J with a single coefficient π_x which is eliminated leaving a single element 1 for the determinant. Thus every row of the determinant which arises from the syzygy has a single element 1. Obviously the rows which arise from equations obtained by multiplying $x_1{}^{m_1}$, $x_2{}^{m_2}$, $\cdot\cdot\cdot$ by individual products π_x have only one element 1 since the x's are eliminated.

It is similarly obvious that the columns of the determinant which arise from the elimination of β_1 and β_2 have a single element 1. The column which arises from the elimination of

$$x_1{}^{k_1}x_2{}^{k_2}x_3{}^{k_3} \qquad (k_1 + k_2 + k_3 = m_1 + m_2 - 1)$$

will if simultaneously

$$0 \leqq k_1 < m_1, \quad 0 \leqq k_2 < m_2, \quad 0 \leqq k_3 < m_2 + r - 1$$

have one element 1 from an equation of the syzygy. If however any one of the three relations,

$$k_1 \geqq m_1, \quad k_2 \geqq m_2, \quad k_3 \geqq m_2 + r,$$

is valid, the column will have one and only one element 1 from one of the three further sets of equations since no two of the above inequalities can hold

simultaneously. Thus the determinant has but one non-zero element in each row and column and is not zero.

In Case IV$_c$ for forms $y_1^{m_1}$, $y_2^{m_1+r}$, $y_3^{m_1+r}$, $y_4^{m_1+r}$ $(r=1,\ 2)$ the syzygy reads:

$$J_{2m_1+2r-3}+\beta_{1y}{}^{m_1+2r-3}\cdot y_1^{m_1}+\beta_{2y}{}^{m_1+r-3}\cdot y_2^{m_2}+\cdots+\beta_{y4}{}^{m_1+r-3}\cdot y_4^{m_1+r}\equiv 0$$

where

$$J_{2m_1+2r-3}=\sum_{k_1,\ldots,k_4} y_1^{k_1}y_2^{k_2}y_3^{k_3}y_4^{k_4}x_1^{m_1-k_1-1}\,x_2^{m_1+r-k_2-1}\cdots x_4^{m_1+r-k_4-1}$$

and

$$k_1+k_2+k_3+k_4=2m_1+2r-3$$

$$0\gtrless k_1<m_1,\quad 0\gtrless k_2<m_1+r,\quad\cdots,\quad 0\gtrless k_4<m_1+r$$

As in the preceding case these restrictions are inconsistent with

$$k_1\geqq m_1,\quad k_2\geqq m_1+r,\quad k_3\geqq m_1+r,\quad k_4\geqq m_1+r$$

and any two of the latter are inconsistent with each other. Hence each equation which arises from the syzygy contributes to a row of the determinant a single element 1 and the same is true of the additional equations.

The columns which arise from the elimination of

$$x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}\qquad(k_1+k_2+k_3+k_4=m_1+m_2-1)$$

will if simultaneously

$$0\gtrless k_1<m_1,\quad 0\gtrless k_2<m_1+r,\quad 0\gtrless k_3<m_1+r,\quad 0\gtrless k_4<m_1+r$$

have just one element 1, an element which arises from J. If however any one of the four mutually exclusive relations,

$$k_1\geqq m_1,\quad k_2\geqq m_1+r,\quad k_3\geqq m_1+r,\quad k_4\geqq m_1+r,$$

is valid, the column again has just one element 1, an element which arises from one of the additional equations. Again the determinant is ± 1 and is not identically zero. This completes the identification of the determinants given in § 2 with the resultants of the corresponding set of forms.

4. *An Identity Connecting $j+1$ General Forms in j Variables.* In scanning the proof of the formula (12) of § 2 the only point at which the assumption, that $\alpha_y{}^m$, $\beta_y{}^n$, $\gamma_y{}^p$, $\delta_y{}^q$ had a common solution x, was used, was in the derivation of (6) where terms in $\alpha_x{}^m,\cdots$ were dropped. If we retain these on the hypothesis that the four forms are general and proceed with the remaining terms as in the derivation of (12) we obtain the formula

$$(27) \qquad \xi_x \cdot J_k \equiv \xi_y \cdot J_{k-1}$$

$$+ (\alpha\beta\gamma\xi) \left[\binom{m-1,\, n-1,\, p-1,\, 0}{k} \cdot \delta_x{}^q - \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} \cdot \delta_y{}^q \right]$$
$$- (\alpha\beta\delta\xi) \left[\binom{m-1,\, n-1,\, 0,\, q-1}{k} \cdot \gamma_x{}^p - \binom{m-1,\, n-1,\, 0,\, q-1}{k-p} \cdot \gamma_y{}^q \right]$$
$$+ (\alpha\gamma\delta\xi) \left[\binom{m-1,\, 0,\, p-1,\, q-1}{k} \cdot \beta_x{}^n - \binom{m-1,\, 0,\, p-1,\, q-1}{k-n} \cdot \beta_y{}^n \right]$$
$$- (\beta\gamma\delta\xi) \left[\binom{0,\, n-1,\, p-1,\, q-1}{k} \cdot \alpha_x{}^m - \binom{0,\, n-1,\, p-1,\, q-1}{k-m} \cdot \alpha_y{}^m \right].$$

If we multiply this by ξ_x and replace on the right $\xi_x \cdot J_{k-1}$ in terms of $\xi_y \cdot J_{k-2}$ by using (27) for $k-1$, and if we continue this process until we have on the left $\xi_x{}^r \cdot J_k$ then in the notation of (14) we shall have

$$(28) \qquad \xi_x{}^r \cdot J_k(\alpha^m, \beta^n, \gamma^p, \delta^q) \equiv \xi_y{}^r \cdot J_{k-r}(\alpha^m, \beta^n, \gamma^p, \delta^q)$$
$$+ J_k(\alpha^m, \beta^n, \gamma^p, \xi^r) \cdot \delta_x{}^q - J_{k-q}(\alpha^m, \beta^n, \gamma^p, \xi^r) \cdot \delta_y{}^q$$
$$- J_k(\alpha^m, \beta^n, \delta^q, \xi^r) \cdot \gamma_x{}^p + J_{k-p}(\alpha^m, \beta^n, \delta^q, \xi^r) \cdot \gamma_y{}^p$$
$$+ J_k(\alpha^m, \gamma^p, \delta^q, \xi^r) \cdot \beta_x{}^n - J_{k-n}(\alpha^m, \gamma^p, \delta^q, \xi^r) \cdot \beta_y{}^n$$
$$- J_k(\beta^n, \gamma^p, \delta^q, \xi^r) \cdot \alpha_x{}^m + J_{k-m}(\beta^n, \gamma^p, \delta^q, \xi^r) \cdot \alpha_y{}^m.$$

If in this we replace $\xi_x{}^r$ by an arbitrary form $\epsilon_x{}^r$ we have the following noteworthy formula, an identity in both x and y which is satisfied by any five quaternary forms:

$$(29) \quad J_k(\alpha^m, \beta^n, \gamma^p, \delta^q) \cdot \epsilon_x{}^r - J_k(\alpha^m, \beta^n, \gamma^p, \epsilon^r) \cdot \delta_x{}^q$$
$$+ J_k(\alpha^m, \beta^n, \delta^q, \epsilon^r) \cdot \gamma_x{}^p - J_k(\alpha^m, \gamma^p, \delta^q, \epsilon^r) \cdot \beta_x{}^n + J_k(\beta^n, \gamma^p, \delta^q, \epsilon^r) \cdot \alpha_x{}^m \equiv$$
$$J_{k-r}(\alpha^m, \beta^n, \gamma^p, \delta^q) \cdot \epsilon_y{}^r - J_{k-q}(\alpha^m, \beta^n, \gamma^p, \epsilon^r) \cdot \delta_y{}^q$$
$$+ J_{k-p}(\alpha^m, \beta^n, \delta^q, \epsilon^r) \cdot \gamma_y{}^p - J_{k-n}(\alpha^m, \gamma^p, \delta^q, \epsilon^r) \cdot \beta_y{}^n + J_{k-m}(\beta^n.\gamma^p, \delta^q, \epsilon^r) \cdot \alpha_y{}^m.$$

For $k=0$, $m=n=p=q=r=1$ this reduces to the usual determinant identity. For $k=0$ alone, it is a long known dual relation which connects five forms and their jacobians.

For $j+1$ forms in j variables,

$$\alpha_{1y}{}^{m_1}, \ \alpha_{2y}{}^{m_2}, \ \cdots, \ \alpha_{j+1,y}{}^{m_{j+1}},$$

the identity is

$$(30) \qquad \sum_{s=1}^{j+1} (-1)^{s+1} \left[J_k(\alpha_1{}^{m_1}, \cdots, \alpha_{s-1}{}^{m_{s-1}}, \alpha_{s+1}{}^{m_{s+1}}, \cdots, \alpha_{j+1}{}^{m_{j+1}}) \cdot \alpha_{sx}{}^{m_s} \right.$$
$$\left. - J_{k-m_s}(\alpha_1{}^{m_1}, \cdots, \alpha_{s-1}{}^{m_{s-1}}, \alpha_{s+1}{}^{m_{s+1}}, \cdots, \alpha_{j+1}{}^{m_{j+1}}) \cdot \alpha_{sy}{}^{m_s} \right] \equiv 0.$$

5. *Alternative Determinantal Expressions for R. The Common Solution when $R=0$; the unique Apolar Form when $R \neq 0$.* That the resultant R may be expressed as a determinant in more than one way, at least in certain

of the above cases, was pointed out by Morley who gave two determinants for the case $m_1 = m_2 = m_3$. We examine this situation more closely. The resultants are obtained from a syzygy containing J_k of order k in y. To bring out the order l in x of J_k we set

$$(31) \qquad J_k \equiv J_{k,l}, \qquad k + l = \sum m_i - j.$$

These orders k, l in x, y respectively were subject to the limitations

$$(32) \qquad k < m_1 + m_2, \qquad l < m_1 + m_2$$

in order to secure, in the first case, the uniqueness of the coefficients β of the syzygy and, in the second case, the linear independence of the additional equations. The elimination of k and l from (31), (32) leads to the inequality (18) of § 2 which was used to separate the cases III, IV, V, VI. Throughout the discussion in § 2 we had used uniformly the maximum value $m_1 + m_2 - 1$ of l.

We denote by

$$(33) \qquad d = | l - k |$$

the *disparity of the syzygy* which contains $J_{k,l}$. As l decreases, k increases so that the maximum value D of d for any one of the cases occurs when k or l is $m_1 + m_2 - 1$. Thus D, *the disparity of the case*, has the value

$$(34) \qquad D = m_1 + m_2 - m_3 - m_4 - \cdots - m_j + j - 2.$$

For the case III with equal orders, and for the cases III_a, III_b, III_c, IV_a, IV_b, IV_c, IV_d, V_a, V_b, VI the values of D are respectively

$$(35) \qquad \begin{array}{l} m_1 + 1, \quad m_1 + r \ (r = 0, 1), \quad m_1 - r + 1 \ (r = 0, 1), \\ m_1 - r + 1 \ (r = 2, \cdots, m_1 + 1), \quad 2 - r \ (r = 0, 1, 2), \quad 0, \\ 2 - r \ (r = 1, 2), \quad 0, \quad 0, \quad 1 - r \ (r = 0, 1), \quad 0. \end{array}$$

We have used in each case of § 2 that syzygy $J_{k,l}$ for which d had the maximum value D for the case. There is however in the early cases at least a considerable range for d from D to 0 or 1 according as D is even or odd. Thus for each value of d $\left(\begin{smallmatrix}0\\1\end{smallmatrix} \lessgtr d \lessgtr D\right)$ there are two syzygies $J_{k,l}$ and $J_{l,k}$ each of which leads to a resultant.

If we examine the two syzygies with leading terms $J_{k,l}$ and $J_{l,k}$ we find that they give rise to resultants whose determinants are the same except for the interchange of rows and columns. Indeed we pass from $J_{k,l}$ to $J_{l,k}$ by interchanging x and y so that the matrix $| \alpha_1, \cdots, \alpha_j |$ of the determinant is transposed. And if we eliminate the coefficients of $\beta_i y^{k-m_i}$ for the one syzygy

in the same order as we introduce additional equations to the other syzygy by multiplying $\alpha_{ix}^{m_i} = 0$ by products $(x)^{k-m_i}$ the rest of the determinants are likewise transposed. We may state then the theorem:

(36) *In each case of* § 2 *of disparity D there are* $\begin{cases} (D/2+1 & (D \text{ even}) \\ (D+1)/2 & (D \text{ odd}) \end{cases}$ *pairs of syzygies, $J_{k,l}$ and $J_{l,k}$, of disparity d $(0 \leqq d \leqq D)$. Each syzygy provides a determinant form of the resultant but the determinants arising from a pair are merely transposed. The orders of these* $\begin{cases} (D+1)/2 \\ (D+1)/2 \end{cases}$ *distinct determinants decrease with d, and for $d = 0$ when D is even the determinant is symmetric.*

From the values of D given in (35) we note that these symmetric cases occur frequently.

Two illustrative examples may be worth while. For three ternary quartics, $m_1 = m_2 = m_3 = 4$, the jacobian is of order 9 and the syzygies available for resultants are $J_{2,7}$, $J_{3,6}$, $J_{4,5}$, $J_{5,4}$, $J_{6,3}$, and $J_{7,2}$. The first three furnish the following forms of R:

$$\begin{array}{c|c|} & 36 \\ 6 & |\alpha_1\alpha_2\alpha_3| \\ 10 & |\alpha_1| \\ 10 & |\alpha_2| \\ 10 & |\alpha_3| \end{array} \; ; \quad \begin{array}{c|c|} & 28 \\ 10 & |\alpha_1\alpha_2\alpha_3| \\ 6 & |\alpha_1| \\ 6 & |\alpha_2| \\ 6 & |\alpha_3| \end{array} \; ; \quad \begin{array}{c|cccc|} & 21 & 1 & 1 & 1 \\ 15 & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\ 3 & |\alpha_1| & & & \\ 3 & |\alpha_2| & & 0 & \\ 3 & |\alpha_3| & & & \end{array} \; ;$$

determinants of orders 36, 28, 24 respectively whose transposed determinants arise from the last three syzygies in reverse order.

For three ternary quintics, $m_1 = m_2 = m_3 = 5$, and jacobian of order 12, the available syzygies are $J_{3,9}$, $J_{4,8}$, $J_{5,7}$, $J_{6,6}$, $J_{7,5}$, $J_{8,4}$, $J_{9,3}$. Four distinct forms of the resultant appear as determinants of order 55, 45, 39, and 37 of which the last is symmetric. They are

$$\begin{array}{c|c|} & 55 \\ 10 & |\alpha_1\alpha_2\alpha_3| \\ 15 & |\alpha_1| \\ 15 & |\alpha_2| \\ 15 & |\alpha_3| \end{array} \; ; \quad \begin{array}{c|c|} & 45 \\ 15 & |\alpha_1\alpha_2\alpha_3| \\ 10 & |\alpha_1| \\ 10 & |\alpha_2| \\ 10 & |\alpha_3| \end{array} \; ; \quad \begin{array}{c|cccc|} & 36 & 1 & 1 & 1 \\ 21 & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\ 6 & |\alpha_1| & & & \\ 6 & |\alpha_2| & & 0 & \\ 6 & |\alpha_3| & & & \end{array} \; ;$$

$$\begin{array}{c|cccc|} & 28 & 3 & 3 & 3 \\ 28 & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\ 3 & |\alpha_1| & & & \\ 3 & |\alpha_2| & & 0 & \\ 3 & |\alpha_3| & & & \end{array} \; .$$

Let $R_{k,l}$ be that determinant form of R which arises from the syzygy

$$J_{k,l} + \beta_{1y}{}^{k-m_1} \cdot \alpha_{1y}{}^{m_1} + \cdot \cdot \cdot + \beta_{jy}{}^{k-m_j} \cdot \alpha_{jy}{}^{m_j} = 0$$

which exists when $R = 0$. If to this syzygy we add the additional term $c \cdot \eta_y{}^k$ (η_y a linear form), then when $R = 0$ the new syzygy still is valid for the value $c = 0$ and arbitrary η. If to the equations obtained from this new syzygy and the original set of additional equations we add the equation, $\xi_x{}^l = 0$, then the new system of linear equations has a determinant $\Delta(\eta^k, \xi^l)$ which is formed from the determinant R by bordering it with ξ^l under, and η^k opposite, the matrix $| \alpha_1, \cdot \cdot \cdot, \alpha_j |$ arising from $J_{k,l}$. When $R = 0$ $\Delta(\eta^k, \xi^l)$ is a form of orders k, l in η, ξ which, for any η, vanishes when and only when $\xi_x = 0$. Hence $\Delta(\eta^k, \xi^l) = \xi_x{}^l \cdot \eta_s{}^k$ where $\eta_s{}^k$ is some form to be determined.

If we start afresh with the syzygy containing $R_{l,k}$ and proceed in the same way by adding a term $c \cdot \xi_y{}^l$ and an additional equation, $\eta_x{}^k = 0$, the eliminant of the system of linear equations is $\Delta'(\xi^l, \eta^k)$ which factors as before into $\eta_x{}^k \cdot \xi_s{}^l$. But the bordered determinant Δ' is merely the bordered determinant with rows D columns interchanged. Hence $\Delta = \Delta'$ and $\xi_x{}^l \cdot \eta_s{}^k = \eta_x{}^k \cdot \xi_{s'}{}^l$, whence $\eta_s{}^k = \eta_x{}^k$ and $\xi_{s'}{}^l = \xi_x{}^l$. Setting $\eta = \xi$ we have $\Delta(\xi^k, \xi^l) = \xi_x{}^{k+l}$. Hence

(37) *If $R_{k,l}$ is one of the determinantal forms of the resultant R and if $R_{k,l}$ be bordered to the right of, and under, the matrix $| \alpha_1, \cdot \cdot \cdot, \alpha_j |$ with respectively variables ξ^k and ξ^l, then the resulting bordered determinant $\Delta(\xi^k, \xi^l)$ is when $R = 0$ the $(k + l)$-th power of ξ_x where x is the common solution.*

We have then a rational integral expression for the common solution x when it exists.

Morley has pointed out that for any j given forms, $\alpha_{iy}{}^{m_i}$, there is a unique apolar form of class $k + l = \sum m_i - j$, say the form, a_ξ^{k+l} i. e. a form for which the j polar forms, $(\alpha_i a)^{m_i} a_\xi^{k+l-m_i}$, vanish identically. Evidently, if $R = 0$, and x is the common solution then $\xi_x{}^{k+l}$ is the common apolar form. Since thus $\Delta(\xi^k, \xi^l)$ is the common apolar form when $R = 0$, and is of lower degree than R in the coefficients of each of the forms $\alpha_{iy}{}^{m_i}$ then $\Delta(\xi^k, \xi^l)$ must be the unique apolar form when $R \neq 0$. Hence

The unique form of order $\sum m_i - j$ apolar to one of the sets of j forms listed in the introduction is given by any one of the bordered resultants $\Delta(\xi^k, \xi^l)$ which pertain to the given set.

6. *A New Version.* We shall obtain some of the above results in another way, in line with Morley's memoir. We consider forms of the same degree. When we wish to pass to those of different degrees we have merely to include symbolic factors. Take $d+2$ numbers $a, b, c, \cdots$ and associate with each a member $\alpha, \beta, \gamma, \cdots$. We are concerned with the sum of homogeneous products or complete symmetric function of degree k,

$$\sum a^k + \sum a^{k-1}b + \sum a^{k-2}b^2 + \cdots$$

which (*loc. cit.*) was called H^k (as in Salmon, *Higher Algebra,* note, p. 290 of the third edition, 1876). We will write it more explicitly as $H_k(a, b, c, \cdots)$; when $a=0$, it becomes $H_k(b, c \cdots)$.

Then we have the formula, for any positive integer k

(1) $$\sum \alpha H_k(b, c, \cdots) = 0$$
provided
(2) $$\sum \alpha = 0 \text{ and } \sum \alpha a = 0.$$

We define H_0 to be 1, and H_{-k} to be 0, so that (1) is true for all integers.

We see that $H_k(a, b, c, \cdots) - a^m H_{k-m}(a, b, c, \cdots)$ contains a only to the power a^{m-1}. Hence $H_k(abc\cdots) - \sum a^m H_{k-m}(abc\cdots)$ loses all powers as high as the mth, provided $2m > k$. Write this truncated form

$$H_k{}^m(a, b, c, \cdots)$$

Write also σ_m for

$$a^m + b^m + c^m + \cdots.$$

Then $\quad H_k{}^m(b, c, \cdots) - \sum(\sigma_m - a^m)\, H_{k-m}(b, c, \cdots) = H_k{}^m(b, c, \cdots)$
whence, so long as $2m > k$,

(3) $$\sum \alpha H_k{}^m(b, c, \cdots) = \sum \alpha a^m H_{k-m}(b, c, \cdots).$$

We now write $\qquad (\alpha x) \mid \beta\gamma \cdots \mid \quad$ for $\alpha, \cdots$

and $\qquad (\alpha y)/(\alpha x) \qquad$ for $a, \cdots$;

these obeying the relations (2).

Take the case of 5 cubic surfaces; these are denoted by $(\alpha x)^3 \cdots (\epsilon x)^3$.

There are 5 jacobians, for instance

$$J(\beta\gamma\delta\epsilon) = \mid \beta\gamma\delta\epsilon \mid (\beta x)^2(\gamma x)^2(\delta x)^2(\epsilon x)^2.$$

$H_1(bcde)$ is $(\beta y)/(\beta x) + (\gamma y)/(\gamma x) + (\delta y)/(\delta x) + (\epsilon y)/(\epsilon x)$

so that $$J H_1 = J_1.$$
Similarly $$J H_2 = J_2.$$
But $$J H_3^3 = J_3.$$

For in forming J_3 we are to take all jacobians of degree 3 in y formed from

$$\begin{array}{ccc} (\beta x)^3, & (\gamma x)^3, & (\delta x)^3,\ (\epsilon x)^3 \\ (\beta x)^2(\beta y), & (\gamma x)^2(\gamma y), & \cdots \\ (\beta x)(\beta y)^2, & (\gamma x)(\gamma y)^2, & \cdots \\ (\beta y)^3, & (\gamma y)^3, & \cdots \end{array}$$

and the last row contributes no jacobian. So

$$J H_4^3 = J_4$$
$$J H_5^3 = J_5$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$J H_8^3 = J_8.$$

Thus J_n is a form in x and y of degree x in y, $8-n$ in x, which is a sum of homogeneous products, but is truncated when $n > 2$. We have then from (1)

$$\Sigma\ (\alpha x) \mid \beta\gamma\delta\epsilon \mid \cdot H_k\left(\frac{(\beta y)}{(\beta x)},\ \frac{(\gamma y)}{(\gamma x)},\ \frac{(\delta y)}{(\delta x)},\ \frac{(\epsilon y)}{(\epsilon x)}\right) = 0$$

or multiplying by $(\alpha x)^2(\beta x)^2(\gamma x)^2(\delta x)^2(\epsilon x)^2$

$$\begin{array}{ll} \Sigma\ (\alpha x)^3 J(\beta\gamma\delta\epsilon) = 0 & \text{when } k = 0 \\ \Sigma\ (\alpha x)^3 J_1(\beta\gamma\delta\epsilon) = 0 & \text{when } k = 1 \\ \Sigma\ (\alpha x)^3 J_2(\beta\gamma\delta\epsilon) = 0 & \text{when } k = 2 \end{array}$$

and we have from (3)

$$\begin{array}{ll} \Sigma\ (\alpha x)^3 J_3(\beta\gamma\delta\epsilon) = \Sigma\ (\alpha y)^3 J(\beta\gamma\delta\epsilon) & \text{when } k = 3 \\ \Sigma\ (\alpha x)^3 J_4(\beta\gamma\delta\epsilon) = \Sigma\ (\alpha y)^3 J_1(\beta\gamma\delta\epsilon) & \text{when } k = 4 \\ \Sigma\ (\alpha x)^3 J_5(\beta\gamma\delta\epsilon) = \Sigma\ (\alpha y)^3 J_2(\beta\gamma\delta\epsilon) & \text{when } k = 5 \end{array}$$

and 3 other formulae, which are these with x, y interchanged. Generally then,

$$\Sigma\ (\alpha x)^3 J_n(\beta\gamma\delta\epsilon) = \Sigma\ (\alpha y)^3 J_{n-3}(\beta\gamma\delta\epsilon)$$

if $J_0 = J$, and $J_{-1} = 0$, $J_{-2} = 0 \cdots$.

The complete list of linear syzygies so obtained may be clearer in another notation. Denote the 5 cubics by

$$Ax^3 \cdots Ex^3$$

the jacobians by

$$
\begin{array}{lll}
 & & BCDEx^8 \cdots \\
J_1(\beta\gamma\delta\epsilon) & \text{by} & BCDEx^7y \\
J_2(\beta\gamma\delta\epsilon) & \text{by} & BCDEx^6y^2
\end{array}
$$

and so on. Then the 12 syzygies are

$$
\begin{aligned}
\sum Ax^3 \cdot BCDEx^8 &= 0 \\
\sum Ax^3 \cdot BCDEx^7y &= 0 \\
\sum Ax^3 \cdot BCDEx^6y^2 &= 0 \\
\sum Ax^3 \cdot BCDEx^5y^3 &= \sum Ay^3 \cdot BCDEx^8 \\
\sum Ax^3 \cdot BCDEx^4y^4 &= \sum Ay^3 \cdot BCDEx^7y \\
\sum Ax^3 \cdot BCDEx^3y^5 &= \sum Ay^3 \cdot BCDEx^6y^2 \\
\sum Ax^3 \cdot BCDEx^2y^6 &= \sum Ay^3 \cdot BCDEx^5y^3 \\
\sum Ax^3 \cdot BCDExy^7 &= \sum Ay^3 \cdot BCDEx^4y^4 \\
\sum Ax^3 \cdot BCDEy^8 &= \sum Ay^3 \cdot BCDEx^3y^5 \\
0 &= \sum Ay^3 \cdot BCDEx^2y^6 \\
0 &= \sum Ay^3 \cdot BCDExy^7 \\
0 &= \sum Ay^3 \cdot BCDEy^8.
\end{aligned}
$$

Since x and y may be interchanged these amount to six only. But for the application to elimination it is important to have the full scheme.

In this application we suppose x to be a common point of By^3, Cy^3, Dy^3, Ey^3 but not on Ay^3. That is we set

$$Bx^3 = Cx^3 = Dx^3 = Ex^3 = 0.$$

We have then

$$
\begin{array}{lrcl}
(1) & BCDEx^8 &=& 0 \\
(2) & BCDEx^7y &=& 0 \\
(3) & BCDEx^6y^2 &=& 0 \\
(4) & Ax^3 \cdot BCDEx^5y^3 &=& \sum^4 By^3 \cdot CDEAx^8 \\
(5) & Ax^3 \cdot BCDEx^4y^4 &=& \sum^4 By^3 \cdot CDEAx^7y \\
(6) & Ax^3 \cdot BCDEx^3y^5 &=& \sum^4 By^3 \cdot CDEAx^6y_2.
\end{array}
$$

We note that hereby J, J_1, $\cdots$, J_5 are contained in the modulus defined by the 4 curves when these curves have a common point. We call the J_i the adjoint linear forms or simply the *adjoints* of the given forms.

To effect the elimination, we may take (4), (5) or (6). With (4), we have the $\binom{8}{3}$ or 56 unknowns $x_1^5, x_1^4x_2, \cdots, x_4^5$, and the 4 unknowns such as $(CDEAx^8)/Ax^3$.

We have since y is arbitrary $\binom{6}{3}$ or 20 equations, and also the 4×10 equations got by raising Bx^3, Cx^3, Dx^3, Ex^3 to quintics. That is we have 60 equations in 60 unknowns.

With (5) we have the 35 unknowns $x_1^4, x_1^3x_2, \cdots, x_4^4$ and the 4×4 coefficients of y in $(CDEAx^7y)/Ax^3$. And we have 35 equations from (5) and 4×4 by raising the cubics to quartics. That is we have 51 equations in 51 unknowns.

With (6) we have $56 + 4$ equations in $20 + 4 \times 10$ unknowns, namely $x_1^3, \cdots, x_4^3$ and the 10 coefficients of y^2 in each of $(CDEAx^6y^2)/Ax^3$.

Numerical Functions of Multipartite Integers and Compound Partitions.

By E. T. Bell.

The multipartite integers and certain of their functions introduced here have partial arithmetics abstractly identical with that section of rational arithmetic (or of the theory of ideals) which is independent of order relations. They have immediate interpretations in terms of the algebra of compound partitions and their functions, here defined apparently for the first time, and they give rise to an extensive new class of properties of such partitions, which in turn can be replaced by sets of restricted diophantine equations. In particular, as a very special instance of the general theorem concerning a new type of arithmetical invariants in § 7, the simpler algebraic aspects of compound partitions are related through these hyper-complex integers to the algebra of the elliptic and theta functions in a manner abstractly identical with the like for a single rational integer; they also exhibit properties which are abstractly identical with the algebra of the numerous arithmetical functions of divisors, such as the totient, Möbius' function, Liouville's function, that originate in the unique factorization law. In short, the new properties of compound partitions unite the consequences of elliptic and Dirichlet processes in rational arithmetic. The more general relations mentioned have no analogues in the usual theory of partitions. The content of the theory is purely algebraic, although the elliptic case furnishes many formulas in a shape well adapted to asymptotic evaluations, to which I hope to return on another occasion.

In this paper we give only the simple general theorems underlying the whole subject. Applications to special instances, such as the elliptic case, are indicated only in sufficient detail to clarify the definitions. The entire theory of elliptic functions can be transposed into sets of relations between what are here called compositions and separations of functions, both of which are purely arithmetical processes, and this is but the simplest instance of the general theory.

The new technical terms necessary have all been chosen so as to recall the corresponding ones to which they are abstractly identical in rational arithmetic.

1. *M-numbers.* Let $(e_1, e_2, e_3, \cdots)$ be a one-rowed matrix formed from the reduced basis $e_1, e_2, e_3, \cdots$ of an abelian group G of any finite or

infinite order. Any element e of G other than the identity is of the form $e_1^{a_1}e_2^{a_2}e_3^{a_3}\cdots$, in which $a_1, a_2, a_3, \cdots$ are rational integers $\geqq 0$ not all zero, and e uniquely determines the matrix $(a_1, a_2, a_3, \cdots)$, which we shall call the *exponent* of e. To the identity we assign the exponent $(0, 0, 0, \cdots)$. We shall be concerned with a G whose elements are identical with their exponents and hence $(e_1, e_2, e_3, \cdots)$ may be ignored.

If c is any finite complex number the symbol $-\infty$ will be defined by the postulates that $-\infty + c = c - \infty = -\infty$, and $-(-\infty)$ does not exist, the last of which is imposed to make a certain type of division meaningless when the divisor is the zero of the set.

We consider the set S of all hypercomplex numbers

$$\alpha_i = (a_{1i}, a_{2i}, \cdots, a_{ki}) \qquad (i = 1, 2, \cdots),$$

in which k is constant but not necessarily finite, and the *coordinates* a_{ri} $(r = 1, \cdots, k)$ are either all finite complex numbers or all equal to $-\infty$, defined by the postulates (Π_j) $(j = 1, 2)$, of which the second is a mere definition.

(Π_1) If α_i, α_j are in S, $\alpha_i = \alpha_j$ only when $a_{ri} = a_{rj}$ $(r = 1, \cdots, k)$.

(Π_2) The *product* $\alpha_i\alpha_j$ of any elements α_i, α_j in S is the element

$$(a_{1i} + a_{1j},\ a_{2i} + a_{2j}, \cdots, a_{ki} + a_{kj}) \text{ of } S.$$

The *special elements* ζ, ϵ of S,

$$\zeta = (-\infty, -\infty, \cdots, -\infty), \quad \epsilon = (0, 0, \cdots, 0)$$

are such that $\zeta\alpha_i = \zeta$, $\epsilon\alpha_i = \epsilon$ for every α_i in S; they are the only elements of S having these respective properties, and hence they are called the *zero, unity* of S.

From the definition of $-\infty$ and (Π_2) it follows that if and only if $\alpha_i \neq \zeta$, there exists in S a unique solution

$$\alpha_x = (a_{1j} - a_{1i},\ a_{2j} - a_{2i}, \cdots, a_{kj} - a_{ki})$$

of $\alpha_i\alpha_x = \alpha_j$, where α_j is any element of S. We write $\alpha_x = \alpha_j/\alpha_i$ and call it the *quotient of* α_j *by* α_i. Hence *if* S' *be* S *with* ζ *omitted,* S' *is an abelian group under multiplication as defined in* (Π_2).

The product of n equal factors α_j in S is written α_j^n.

The elements of S are called *multipartite numbers of order* k or, since k is constant in a given context, briefly *M-numbers.*

We segregate M-numbers into classes according to the customary classification of complex numbers. An M-number is said to be *rational* if each of

its coordinates is a rational integer $\geqq 0$; otherwise it is *irrational.* A rational M-number is called an *M-integer* if each of its coordinates is $\geqq 0$; otherwise it is an *M-fraction.* An irrational M-number is *real* or *imaginary* according as none or at least one of its coordinates is of the form $a + \pi i/2$, where a is real and π, i here (but not later) have their usual meanings. This classification is merely a restatement that M-numbers initially are exponents.

We shall be concerned here only with M-integers, that is with the set of all $(a_1, \cdots, a_k)$ subject to (Π_j) $(j = 1, 2)$, where the a_r $(r = 1, \cdots, k)$ range independently over all rational integers $\geqq 0$.

If α_i, α_j are M-integers, the quotient α_i/α_j is in general not an M-integer. The distinction thus introduced between algebraic and arithmetic divisibility is the source of interest of the theory.

2. *Divisibility for M-integers.* We say that the M-integer α *divides* the M-integer β, and write $\alpha \mid \beta$, if and only if an M-integer γ exists such that $\beta = \alpha\gamma$. If $\alpha = (a_1, \cdots, a_k)$, $\beta = (b_1, \cdots, b_k)$, the necessary and sufficient conditions that $\alpha \mid \beta$ are $b_r \geqq a_r$ $(r = 1, \cdots, k)$.

If $\delta \mid \alpha$ and $\delta \mid \beta$, δ is a *common divisor* of α, β. If δ is a common divisor of α, β, and if every common divisor of α, β divides δ, δ is called a G.C.D. of α, β. If $\alpha \mid \mu$, $\beta \mid \mu$, μ is a *common multiple* of α, β. If μ is a common multiple of α, β, and if every common multiple of α, β is a multiple of μ, μ is called a L.C.M. of α, β. It follows from these definitions that α, β have precisely one G.C.D. and precisely one L.C.M., and that if α, β are as above, the G.C.D. of α, β is $(d_1, \cdots, d_k)$ where $d_r =$ the lesser of a_r, b_r (or either of these if $a_r = b_r$) for $r = 1, \cdots, k$, and the L.C.M. is $(m_1, \cdots, m_k)$ where m_r is the greater of a_r, b_r $(r = 1, \cdots, k)$.

If the G.C.D. of a set of M-integers is the unity, ϵ, the members of the set are called *coprime.* An M-integer $\pi \neq \epsilon$ whose only divisors are π, ϵ is called an *M-prime.* Hence all the M-primes are the k M-integers.

$$(1, 0, 0, \cdots, 0),\ (0, 1, 0, \cdots, 0),\ \cdots,\ (0, 0, 0, \cdots, 1),$$

which will be denoted by $\pi_1, \pi_2, \cdots, \pi_k$ respectively. An arbitrary M-prime will be denoted by π.

If $\pi \mid \alpha\beta$ then at least one of $\pi \mid \alpha$, $\pi \mid \beta$ is true. From this and the uniqueness of the G.C.D. it follows as usual that *an M-integer $\alpha \neq \epsilon$ is uniquely the product of M-primes,* except for unit factors (which may be ignored). If $\alpha = (a_1, a_2, \cdots, a_k)$ the resolution of α into its prime M-factors is

$$(1) \qquad \alpha = \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_k^{a_k},$$

and α has precisely the $\nu(\alpha) = (a_1 + 1) \cdots (a_k + 1)$ divisors

$$(d_1, \cdots, d_k), \quad 0 \leqq d_j \leqq a_j \qquad (j = 1, \cdots, k).$$

From the resolutions of several M-integers into their prime factors we can write down the G.C.D. and L.C.M. of the set as in rational arithmetic. Evidently we might have started from (1) and thence have obtained all the preceding definitions concerning M-integers, but it is of interest to follow the abstract identity of M-integers and rational integers directly from the postulates.

The connection with compound partitions is as follows. Let μ, α, β, $\cdots$, γ be M-integers $\neq \epsilon$ such that

$$\mu = \alpha^a \beta^b \cdots \gamma^c \tag{2}$$

where $a, b, \cdots, c$ are rational integers $\geqq 0$, and

$$\alpha = (a_1, \cdots, a_k), \ \beta = (b_1, \cdots, b_k), \cdots, \gamma = (c_1, \cdots, c_k),$$
$$\mu = (m_1, \cdots, m_k).$$

Then (2) is equivalent to

$$m_j = aa_j + bb_j + \cdots + cc_j \qquad (j = 1, \cdots, k),$$

which is a compound partition of the set of integers m_j $(j = 1, \cdots, k)$ if these be regarded as constants. The significance of unique factorization of M-integers in terms of compound partitions will appear as we proceed.

3. *Separations mod C of M-integers.* Let C be a set of rational integers > 0; C may contain any finite number or an infinity of distinct elements (rational integers), but no element may occur more than once in C. The last is not a restriction on the generality of the processes next considered, as will be evidenced later. If C contains the integer $n > 0$, we write $C \mid n$.

Corresponding to the partitions in the usual sense of a single rational integer we shall be concerned here with the separations modulo C of M-integers, where C is a given set as above, which are defined as follows. Let α_i $(i = 1, \cdots, s)$ be any s M-integers all different from ϵ such that, for α a given M-integer,

$$\alpha = \alpha_1^{c_1} \alpha_2^{c_2} \cdots \alpha_s^{c_s}$$

where $C \mid c_i$ $(i = 1, \cdots, s)$, *and no two of the* c_i *are equal.* Indicating the set C which contains the c_i we write the above in the form

$$\alpha = \alpha_1^{c_1} \alpha_2^{c_2} \cdots \alpha_s^{c_s} \pmod{C} \tag{3}$$

when necessary. The right of (3) is called a *separation mod C of order s of* α, and the $s\,!$ separations obtained by permuting the factors $\alpha_i^{c_i}$ $(i=1,\cdots,s)$ in all possible ways are defined to be identical. When α is given and s is constant the number of separations mod C of α of order s is finite, and the total number of separations of all orders $s=1,2,3,\cdots$ mod C of α is finite. The set of all separations mod C of α of all orders is called the *total separation mod C of* α. For every C we assign by convention to ϵ the total separation ϵ. An example of (3) is given near the end of § 5. It does not follow that for C, α given a separation of α (mod C) exists.

To state the equivalent of (3) in terms of compound partitions, disregard the trivial case $\alpha=\epsilon$, and let

$$\alpha_i=\pi_1^{a_{1i}}\pi_2^{a_{2i}}\cdots\pi_k^{a_{ki}} \qquad (i=1,\cdots,s)$$

be the resolutions of the α_i into their M-prime factors. Since in (3) $\alpha_i\neq\epsilon$ $(i=1,\cdots,s)$, the α_{ji} are ks rational integers $\geqq 0$ such that

$$a_{1i}+a_{2i}+\cdots+a_{ki}>0 \qquad (i=1,\cdots,s).$$

Let α be as in (1). Then, comparing exponents of π_j in (3) we see that (3) is equivalent to the restricted system of k diophantine equations

$$a_i=\sum_{j=1}^{s} c_j\,a_{ij}, \qquad C\mid c_j \qquad (i=1,\cdots,k),$$

where the a_i $(i=1,\cdots,k)$ are any given k integers $\geqq 0$ whose sum is >0, s is constant, and the c_j, a_{ij} are to be determined subject to the stated conditions.

4. *Numerical functions of M-integers.* If $f(\alpha)$ takes a single finite value when α is any given M-integer, and if $f(\epsilon)=1$, we call $f(\alpha)$ a *numerical function* of α. The *value* of $f(\alpha)$ is to be understood in the usual sense as a definite real or complex number. Hence the value of $f(\alpha)$ is not an M-number, and we operate simultaneously in two domains, that of M-integers and that of real or complex numbers.

The following instances of numerical functions will be useful later. Let $a,b,\cdots,c$ be κ distinct integers chosen from $1,2,\cdots,k$, and let

$$\delta=\pi_a^{p}\pi_b^{q}\cdots\pi_c^{r} \qquad (p,q,\cdots,r>0)$$

be the resolution of δ into its M-prime factors. Write

(4) $\qquad \rho(\delta)=p+q+\cdots+r, \quad \gamma(\delta)=\kappa;$

(5) $\qquad \lambda(\delta)=(-1)^{\rho(\delta)}, \quad \lambda(\epsilon)=1;$

(6) $\qquad \mu(\epsilon)=1, \quad \mu(\delta)=0$ if at least one of $p,q,\cdots,r>1$,

and otherwise $=(-1)^{\gamma(\delta)}$. Then $\lambda(\alpha)$, $\mu(\alpha)$ are numerical functions of α, while $\rho(\alpha)$ is not, since $\rho(\epsilon)=0$.

If $f(\alpha\beta)=f(\alpha)f(\beta)$ for every pair of coprime M-integers α, β, we call $f(\alpha)$ a *normal* function of α. Hence, *if f is normal and δ as above,*

$$f(\delta)=f(\pi_a{}^p)f(\pi_b{}^q)\cdots f(\pi_c{}^r).$$

If $f(\alpha)=g(\alpha)$ for all M-integers α, we say that $f(\alpha)$, $g(\alpha)$ are *equal,* and write $f=g$.

5. *Separations mod C of numerical functions.* Let f be any numerical function and α a constant M-integer. Form the product $f(\alpha_1)f(\alpha_2)\cdots f(\alpha_s)$ with respect to the separation (3), and indicate by $\sum_s$ a sum with respect to all orders $s=1,2,3,\cdots$, that is, over the total separation of α. Then the finite sum $f'(\alpha)$ defined by

$$f'(\alpha)=\sum_s f(\alpha_1)f(\alpha_2)\cdots f(\alpha_s) \tag{7}$$

is a numerical function of α, which we shall call the *separation mod C of* $f(\alpha)$ or, when C is understood, the *separation* of $f(\alpha)$. If necessary to indicate C we may write $f'(\alpha)$ (mod C). A separation will always be indicated by an accent, as in the left of (7).

If $\alpha=\pi^n$, $n>0$, where as always π is any M-prime, $f'(\alpha)$ (mod C) is connected as follows with certain of the partitions of n, as is seen at once from the definitions. Let

$$n=j_1n_1+j_2n_2+\cdots+j_sn_s$$

be any partition of n into s unequal integers n_r $(r=1,\cdots,s)$ in C, repeated $j_r>0$ times respectively. As in the classic theory of partitions the sequence of the parts in a given partition is immaterial; thus $7+35$ and $35+7$ are the same partition of 42 in the set of all integer multiples >0 of 7. As in a frequent notation we write the above partition in the form

$$n=(n_1{}^{j_1}n_2{}^{j_2}\cdots n_s{}^{j_s}) \qquad (\text{mod } C), \tag{8}$$

where mod C refers to $n_1, n_2, \cdots, n_s$. We call (8) a *partition mod C of order s of n.* Let $\sum_s$ indicate a sum with respect to all partitions mod C of all orders $s=1,2,3,\cdots$ of n. Then, the separation (7) when $\alpha=\pi^n$ is easily seen to be

$$f'(\pi^n)=\sum_s f(\pi^{j_1})f(\pi^{j_2})\cdots f(\pi^{j_s}). \tag{9}$$

By convention, for every mod C, we assign to 0 the unique partition 0, and take

$$f'(\pi^0) = f'(\epsilon) = f(\epsilon) = 1.$$

If f is normal, and is as in (1),..

$$f'(\alpha) = f'(\pi_1^{a_1})\, f'(\pi_2^{a_2}) \cdots f'(\pi_k^{a_k}), \tag{10}$$

which can be computed by (9).

As a numerical example, let C for the moment be the set of all rational integers > 0, and consider the separation $f'(p^2q^3)$ (mod C) of $f(p^2q^3)$, where p, q are M-primes. Then $\alpha = p^2q^3$ has only the six separations, as in (3),

$$\begin{array}{lll} (p)^2(q)^3, & (pq)^2(q)^1, & (p^2q^3)^1, \\ (p^2)^1(q)^3, & (p^2q)^1(q)^2, & (p)^2(q^3)^1, \end{array}$$

of the respective orders 2, 2, 1, 2, 2, 2, the M-integers in parentheses being the $\alpha_1, \alpha_2, \cdots$ of (3) and the exponents outside parentheses the $c_1, c_2, \cdots$. Hence, by (7) the value of $f'(p^2q^3)$ (mod C) is

$$f(p)f(q) + f(pq)f(q) + f(p^2q^3) + f(p^2)f(q) \\ + f(p^2q)f(q) + f(p)f(q^3).$$

If in particular f is normal (but not in general otherwise),

$$f(pq)f(q) = f(p)f^2(q), \quad f(p^2q^3) = f(p^2)f(q^3), \\ f(p^2q)f(q) = f(p^2)f^2(q),$$

where $f^2(q) = [f(q)]^2$, and so in all like cases. Hence, for normal f, we have from the above value of $f'(pq)$ for any f,

$$f'(p^2q^3) = [f(p) + f(p^2)]\,[f(q) + f^2(q) + f(q^3)].$$

Again, corresponding to (8) the partitions mod C for the exponents 2, 3 in p^2q^3 are

$$2 = (2^1) = (1^2), \quad 3 = (3^1) = (2^1 1^1) = (1^3),$$

of the respective orders 1, 1, 1, 2, 1. Hence, by (9), we have the separations mod C

$$f'(p^2) = f(p) + f(p^2), \quad f'(q^3) = f(q) + f^2(q) + f(q^3).$$

Thus, if f is normal,

$$f'(p^2q^3) = f'(p^2)\, f'(q^3) \quad \text{mod } C,$$

in accordance with the theorem, and we might have computed $f'(p^2q^3)$ for normal f directly from $f'(p^2)$, $f'(q^3)$ with less labor. It is clear that for any other C than that of the above examples the separations will differ from those just found, also it is evident that the separations $f'(p^2)$, $f'(q^3)$ (mod C) do not enable us to calculate $f'(p^2q^3)$ mod C unless f is normal, and so in all cases. Separations may exist for one mod C but not for another; for any mod C there obviously are an infinite number of $f(\alpha)$ which have separations.

6. *Compositions of numerical functions.* Let α_j $(j=1,\cdots,r)$ be any r M-integers whose product is the constant M-integer α,

$$\alpha=\alpha_1\alpha_2\cdots\alpha_r. \tag{11}$$

Note that we have not excluded the case, which will always be relevant in what follows, when certain of the α_j may be ϵ. Attending to the arrangement of the factors $\alpha_1, \alpha_2, \cdots, \alpha_r$, we call the right of (11) a *composition of order r of n.* Thus, if $\alpha=\alpha_1\alpha_2$, and $\alpha_1\neq\alpha_2$, the compositions $\alpha_1\alpha_2$, $\alpha_2\alpha_1$ of order 2 of α are distinct, as also are $\epsilon\alpha$ and $\alpha\epsilon$.

Let f, f_i $(i=1,\cdots,r)$ be any $r+1$ numerical functions. For α, r constant, the finite sum

$$\sum f_1(\alpha_1)\,f_2(\alpha_2)\cdots f_r(\alpha_r) \tag{12}$$

taken over all compositions of α of order r, is evidently a numerical function of the single M-integer α, say $f(\alpha)$. We call this $f(\alpha)$ a *composition of order r.* Clearly $f(\alpha)$ is uniquely known when α and f_i $(i=1,\cdots,r)$ are assigned. Conversely, as will appear presently, if f be given it is possible in an infinity of ways to determine, for r constant, r numerical functions f_i $(i=1,\cdots,r)$ such that, for every α, $f(\alpha)$ is equal to the sum (12). Each such determination of the f_i $(i=1,\cdots,r)$ for a given f is called a *composition of order r of f,* and we write

$$f(\alpha)=\sum f_1(\alpha_1)\,f_2(\alpha_2)\cdots f_r(\alpha_r), \tag{13}$$

or, omitting all arguments and the sign of summation,

$$f=f_1f_2\cdots f_r, \tag{14}$$

as the symbolic equivalent of this composition. The $f_1, f_2, \cdots, f_r$ are called *factors* of f, and (14) defines a species of multiplication, concerning which we have the following theorem.

Composition as in (14) *is associative and commutative.*

Let $e=e(\alpha)$ be that numerical function which is defined by

$$e(\epsilon)=1, \quad e(\alpha)=0, \quad \alpha \neq \epsilon. \tag{15}$$

We call e the *unit function of composition,* or briefly the *unit function,* since $ef=f$ for every numerical function f.

If g is any numerical function there exists a unique numerical function, denoted by g^{-1}, and called the reciprocal of g, such that $gg^{-1}=e$.

The proof, abstractly identical with that of a former paper * for functions of rational integers, incidentally furnishes the explicit form of $g^{-1}(\alpha)$ in terms of functions g with the appropriate arguments determined by α. It need not be reproduced.

From the existence of reciprocals that of an infinity of compositions (14) of a given f as already stated is obvious.

The set of all numerical functions is an abelian group under composition, the identity of the group being e.

As a detail of notation it is necessary to distinguish between the composition of r equal functions and their algebraic product, which also can be taken as a single numerical function (when the arguments of all the factors are identical) and hence can occur as a factor in a composition. If in (14) $f_i=g$ $(i=1,\cdots,r)$, we shall write (14) as $f=g^r$. If on the other hand the algebraic product $[g(\alpha)]^r$ is to be represented with the argument α omitted, and hence in a form suitable for composition as in (14), we shall write the product as (g^r). For example, to illustrate both, the composition $h=f^2(g^3)$ signifies, for every M-integer α, the equality

$$h(\alpha)=\sum f(\alpha_1)\,f(\alpha_2)\,g(\alpha_3)\,g(\alpha_3)\,g(\alpha_3),$$

where $\sum$ is taken over all sets $(\alpha_1, \alpha_2, \alpha_3)$ of solutions $\alpha_1, \alpha_2, \alpha_3$ of $\alpha=\alpha_1\alpha_2\alpha_3$.

7. *Separation mod C and composition of numerical functions.* Combination of the stated processes leads to the most interesting properties of M-integers, particularly when the theorems, abstractly identical with those properties of rational integers or of ideals which depend only upon the unique factorization law of arithmetic, are restated as their equivalents in terms of compound partitions. All of the properties of compound partitions thus obtained appear to be new.

Abstractly, what follows is equivalent algebraically to the simultaneous performance of Dirichlet multiplication and Dedekind inversion in an en-

* *Tohoku Mathematical Journal,* Vol. 17 (1920), pp. 221-231.

larged sense, and the expansion of infinite products whose simplest instances are those occurring in elliptic functions. It thus combines, in its simplest instance, the multiplicative processes of algebraic arithmetic as given by operations upon Dirichlet series and those of the additive that are contained in relations between elliptic and theta functions. To composition correspond the Dirichlet processes, to separation the elliptic and their generalizations; the following combines them.

Let $f = f_1 f_2 \cdots f_r$ be any composition of order r of f. Then

$$f' = (f_1 f_2 \cdots f_r)' = f_1' f_2' \cdots f_r'; \tag{16}$$

that is, the separation of a composition is the composition of the separations of the several factors of the composition. Further, the relation (16), *implied by $f = f_1 f_2 \cdots f_r$, is the same for all sets C, and hence is a species of invariant of C.* This will be evident from the next section.

Thus far we have considered separations with respect to a single set C. Frequently, however, it is necessary in applications to discuss simultaneously separations with respect to several sets. Let C_j $(j = 0, 1, 2, \cdots)$ be any sets defined in the same way as C, and write $\phi_j'(\alpha)$ for the separation $\phi'(\alpha)$ (mod C_j), where $\phi(\alpha)$ is any numerical function of α. Then ϕ'_j $(j = 0, 1, \cdots)$ being numerical functions, the ϕ_j' can be taken as factors in compositions, and the entire preceding theory of composition of numerical functions applies to them. Similarly, if ψ, χ, $\cdots$ are any numerical functions, $\psi_j' \chi_j' \cdots$ $(j = 0, 1, \cdots)$ are defined, and all of $\phi_j', \psi_j', \chi_j', \cdots$ $(j = 0, 1, \cdots)$, or any subset of them, are subject to composition. We thus operate simultaneously in sets of sets as moduli, in analogy with the familiar modular systems of algebraic numbers. If all of the $\phi_j', \psi_j', \chi_j', \cdots$ considered in a given context are normal, there is the obvious extension of (10) available for calculations. Examples will be found in § 11.

8. *Generators.* It is assumed as usual that indeterminates can be combined according to the processes of an abstract field, and that equality of polynomials or power series in several indeterminates signifies only that the coefficients of like power products of indeterminates are equal. As in any use of indeterminates in the literature of arithmetic, their introduction here is merely a convenience which can be avoided if desired. The suppression of the indeterminates however contributes nothing to the logic of the situation.

If t is a parameter, π any M-prime, α any M-integer, we treat as indeterminates the powers π^t, α^t which, by hypothesis, are subject to all the laws

of common algebra. In particular, if α, β are any M-integers, $\alpha^t\beta^t = (\alpha\beta)^t$; $(\alpha^t)^n = \alpha^{nt}$, where $n > 0$ is a rational integer;

$$(\alpha^t)^0 = \alpha^0 = \epsilon.$$

For convenience, we now write $\pi^t = z$, $\alpha^t = w$, where π, α are the symbols of a general M-prime and a general M-integer; $\sum_\alpha$ refers to a sum over all M-integers, $\prod_\pi$ to a product over all M-primes, and $\prod_n \pmod{C}$ to a product over all n such that $C \mid n$, where C is any given set. The result of replacing t by nt is the same as that of changing z, w to z^n, w^n respectively.

If $f(\alpha)$, $g(\alpha)$ are any numerical functions, an equation

$$\sum_\alpha f(\alpha)w = \sum_\alpha g(\alpha)w$$

is equivalent to $f(\alpha) = g(\alpha)$ for all M-integers α, and hence to $f = g$. It is convenient always to take as the first term of any $\sum_\alpha$ that in which $\alpha = \epsilon$; the sequence of the remaining terms is entirely arbitrary.

Let several such series $\sum_\alpha$ be multiplied together. Regroup the distributed product as a new $\sum_\alpha$, that is, as a series in which the total coefficient of a given M-integer α is collected as that of the corresponding $w = \alpha^t$. The product evidently is unique. Hence, *the set of all $\sum_\alpha f(\alpha)w$, as f runs through the set of all numerical functions, is closed under such multiplication of series $\sum_\alpha$.*

Let $f = f_1f_2 \cdots f_r$ as in (14). Then

$$\sum_\alpha f(\alpha)w = \sum_\alpha f_1(\alpha)w \times \sum_\alpha f_2(\alpha)w \times \cdots \times \sum_\alpha f_r(\alpha)w. \tag{17}$$

Again, if g be any numerical function, we see that

$$\prod_n \left[\sum_\alpha g(\alpha)w^n\right] = \sum_\alpha g'(\alpha)w \pmod{C}, \tag{18}$$

where as always $g'(\alpha)$ is the separation $\pmod{C}$ of $g(\alpha)$.

If h be any numerical function, then

$$\prod_\pi \left[1 + h(\pi)z + h(\pi^2)z^2 + \cdots\right] = \sum_\alpha h(\alpha)w. \tag{19}$$

The series in [ ] on the left may be finite or infinite.

If g, g^{-1} are reciprocals, then

$$\sum_\alpha g(\alpha)w \times \sum_\alpha g^{-1}(\alpha)w = 1. \tag{20}$$

If h is as in (19), then

$$(21) \qquad [1 + h(\pi)z + h(\pi^2)z^2 + \cdots] \\ [1 + h^{-1}(\pi)z + h^{-1}(\pi^2)z^2 + \cdots] = 1,$$

in which, as always, h^{-1} is the reciprocal of h as defined in § 6.

In (17) replace w by w^n, where $C \mid n$. Then letting n range over all elements of C, we have

$$\Pi_n [\Sigma_a f(\alpha) w^n] = \Pi_n [\Sigma_a f_1(\alpha) w] \times \cdots \times \Pi_n [\Sigma_a f_r(\alpha) w^n],$$

which is equivalent to

$$\Sigma_a f'(\alpha) w = \Sigma_a f_1'(\alpha) w \times \cdots \times \Sigma_a f_r'(\alpha) w.$$

Hence, if for the moment we write $k = f_1' f_2' \cdots f_r'$, we have

$$\Sigma_a f'(\alpha) w = \Sigma_a k(\alpha) w,$$

which is equivalent to the theorem in § 7.

Returning now to (19) and h normal as there, we shall call the typical factor

$$(22) \qquad H(z) = 1 + h(\pi)z + h(\pi^2)z^2 + \cdots$$

the *generator* of $h(\alpha)$, or of h. Hence *the generator of the composition of any number of normal numerical functions is the product of the generators of the several functions; from* (21), *the generator of the reciprocal of a normal numerical function is the reciprocal of the generator of the function.*

From (19), (22) we have

$$\Pi_n [\Pi_\pi H(z^n)] = \Pi_\pi [\Pi_n H(z^n)] = \Sigma_a h'(\alpha) w \pmod{C};$$

and hence, if

$$\Pi_n H(z) = 1 + g(\pi)z + g(\pi^2)z^2 + \cdots \equiv G(z),$$

we get $g = h'$, and $G(z)$ is the generator of h'. If $H(z)$ contains more than 2 terms the explicit determination of $g(\pi^j)$ $(j = 1, 2, \cdots)$ may be difficult. When $H(z)$ contains only 2 terms, the least number possible, we can apply the known elliptic products in a great many instances.

If as in (17) none of $f_1, \cdots, f_r$ are necessarily normal, the generator of $f_j(\alpha)$ is defined to be $\Sigma_a f_j(\alpha) w$, and we have precisely the same conclusion as in the foregoing theorem. The case of normal functions is however that of greater interest.

In the definition of numerical functions in § 4 it is not necessary to

impose the condition $f(\epsilon) = 1$, which was done merely for simplicity; it is sufficient to take $f(\epsilon) \neq 0$. The consequent modifications of all that precedes can be readily made.

Finally, as in § 7, end, all of this section has an immediate extension to separations with respect to any set of sets C_j $(j = 0, 1, \cdots)$. The generators corresponding to different sets $C_0, C_1, \cdots$ may be combined by multiplications and divisions, in several ways. Thus, as the simplest, only those generators appertaining to a given C_j may be combined. Alternatives will be noticed in § 12 after the examples in § 11.

9. *Classes of normal functions and M-integers.* Normal functions are now classified as algebraic or transcendental, in a technical sense, according to their generators considered as functions of two independent indeterminates π, z. The normal h is called *algebraic* or *transcendental* according as its generator (22) is an algebraic or a transcendental function of π, z.

If algebraic, h is defined to be of the same species as its generator, for example rational, rational integral, etc. The generator of a rational algebraic h has a *canonical form* $P(\pi, z)/Q(\pi, z)$, in which P, Q are polynomials in π, z with no common factor other than 1, and the leading coefficients in P, Q are both 1; the case in which one but not both of P, Q is 1 is included.

If $Q = 1$ in the above, the normal h generated by $P(\pi, z)$ is called *prime* or *composite* according as $P(\pi, z)$ is irreducible or reducible in the rational domain. Hence, *an algebraic h is uniquely of the form*

$$g_1 g_2 \cdots g_r \quad k_1 k_2 \cdots k_s \tag{23}$$

in which g_i $(i = 1, \cdots, r)$ *are prime functions,* k_j $(j = 1, \cdots, s)$ *are reciprocals of prime functions, no* k_j *is the reciprocal of any* g_i, *and either all the* g_i *or all the* k_j *may be absent.* We call (23) the *prime composition* of h; it is significant here only if h be algebraic as defined, although it may obviously be extended to certain transcendental h.

To apprehend readily the structure of a given normal function it is useful to separate all M-integers into two classes, simple and non-simple. Exactly the same principle applies to any set of elements having a unique factorization law, and hence in particular to the rational integers. In the latter case we may indeed take $\pi_k =$ the kth natural prime; M-integers then become rational integers > 0.

An M-integer is called *simple* if and only if it is divisible by the square of no M-integer other than the unity ϵ. We shall exclude ϵ from the simple

M-integers, which therefore are of the form $(j_1, j_2, \cdots, j_k)$ where each of j_i $(i = 1, \cdots, k)$ is a definite one of 0, 1, and not all are 0. There are thus $2^k - 1$ simple M-integers if k be finite. A set of simple M-integers every pair of which are coprime are called *distinct.*

From the definitions it follows that *any M-integer* $\alpha \neq \epsilon$ *has a unique resolution into a product of powers of distinct simple M-integers* σ_j $(j = 1, \cdots, r)$ *of the type*

$$(24) \qquad \alpha = \sigma_1^{s_1} \sigma_2^{s_2} \cdots \sigma_r^{s_r},$$

in which the s_j $(j = 1, \cdots, r)$ *are rational integers* > 0 *no two of which are equal.*

In the generator (22) of any normal h certain of the coefficients $h(\pi^j)$ $(j = 1, 2, \cdots)$ may be identically zero. Retaining only non-vanishing coefficients we rewrite the generator of h as

$$(25) \qquad 1 + h(\pi^{j_1})z^{j_1} + h(\pi^{j_2})z^{j_2} + \cdots,$$

which may be either a finite or an infinite series; if h is a prime function, (25) is a polynomial. The h generated by (25) is completely defined by the following properties. Let (24) be the resolution of α into distinct simple factors. Then, if any one of s_i $(i = 1, \cdots, r)$ is not a member of the sequence $j_1, j_2, \cdots$, the function $h(\beta)$ vanishes for $\beta = \alpha$. To state the value of $h(\alpha)$ in the contrary case, let

$$\sigma_i = \pi_{i1} \pi_{i2} \cdots \pi_{im}, \quad m \equiv k_i \qquad (i = 1, \cdots, r).$$

Then, by (24), the $k_1 + k_2 + \cdots + k_r$ M-primes π_{ij} are all different. Write

$$H(\sigma_i) = h(\pi_{i1}^{s_1}) \, h(\pi_{i2}^{s_2}) \cdots h(\pi_{im}^{s_m}) \qquad (i = 1, \cdots, r).$$

Then the required value is

$$h(\alpha) = H(\sigma_1) \, H(\sigma_2) \cdots H(\sigma_r).$$

When $h(\alpha)$ is a prime algebraic function its generator (25) contains only a finite number n of distinct powers z^{j_i} $(i = 1, \cdots, n)$, and likewise for the product of any finite number of equal or distinct prime functions. The number of distinct powers of z in such a finite generator is called its *extent;* the extent of a function (when significant) is defined to be the *extent of its generator.* The simplest algebraic numerical functions are those of extent 1 and their reciprocals.

To investigate the algebraic relations between a given set of algebraic

numerical functions we decompose their respective generators into their irreducible factors, thereby obtaining the corresponding resolutions (23), and from the latter, by multiplications and divisions as in any abelian group, write down any number of relations between compositions of functions, in abstract identity with the set of all relations derived by multiplications and divisions in the set of all rational numbers. From these relations we can pass at once as in § 7 (16) to abstractly identical relations between separations. Finally, as noted in § 3, any such relation is equivalent to a theorem in compound partitions. All the relations in the foregoing are evidently implicit in the prime resolutions (23). By obvious modifications we get similar results when the functions are transcendental. Irrational algebraic numerical functions are treated in an evident way by means of their generators regarded as power series in z.

10. *Addition and subtraction.* As in all arithmetical theories (rational arithmetic, ideals, etc.), there is here a sharp break between the properties of multiplication for M-numbers and those of addition as next defined. It seems in fact as if arithmetic is essentially two distinct theories, and that only adventitious circumstances (in rational arithmetic, but not strictly in the theory of ideals or even in that of algebraic integers without ideals) permit the two theories to be joined into one. That the nature of these circumstances is not fully understood seems fairly clear from the inordinate difficulties encountered in attempts to prove such simple conjectures as that which asserts every even integer to be the sum of two primes. The properties of M-integers as so far investigated have grown out of multiplication and its inverse division. From the above remarks it is reasonable to expect nothing natural when we introduce addition, which is now done merely for completeness. It is openly *ad hoc* (like most attempts to define addition for ideals).

Addition and its inverse subtraction for M-numbers are defined by the postulate that polynomials in M-numbers with coefficients in any field are combined as if the M-numbers were indeterminates. Hence equality of functions of M-numbers taken over any field is defined. Those processes are of but slight interest; the next are more significant.

Since the values of numerical functions are ordinary real or complex numbers, the values form a field. Let $f(\alpha)$ be any numerical function of the M-integer α, and write $f''(\alpha)$ for its value. Then if $a, b, \cdots, c$ are any real or complex numbers, $f(\alpha), g(\alpha), \cdots, h(\alpha)$ any numerical functions,

$$F''(\alpha) \equiv af''(\alpha) + bg''(\alpha) + \cdots + ch''(\alpha)$$

is defined as α runs through all M-integers. We say that $F''(\alpha)$ is the *value* of the *extended numerical function*

$$F(\alpha) \equiv af(\alpha) + bg(\alpha) + \cdots + ch(\alpha),$$

or, dropping the argument α as before, we define the extended numerical function F by

$$F = af + bg + \cdots + ch,$$

and this F is merely the symbol for the function which takes the values $F''(\alpha)$ as α runs through all M-integers α. It is clear that F is not in general a numerical function as defined in § 4, as is seen from the example

$$a + b + \cdots + c = 0, \quad f = g = \cdots = h.$$

On the other hand extended numerical functions include numerical functions as a special case; it suffices to take

$$a = 1, \quad b = \cdots = c = 0.$$

Let

$$F = af + bg + \cdots + ch, \quad G = p\phi + q\psi + \cdots + r\chi$$

be any extended numerical functions, $f, g, \cdots, h, \phi, \psi, \cdots, \chi$ being numerical functions, and $a, b, \cdots, c,\ p, q, \cdots, r$ real or complex numbers. The *product* FG of F, G is the extended numerical function

$$\begin{aligned} & apf\phi + aqf\psi + \cdots + arf\chi \\ & + bpg\phi + bqg\psi + \cdots + brg\chi \\ & + \cdots\cdots \\ & + cph\phi + cqh\psi + \cdots + crh\chi, \end{aligned}$$

in which $f\phi, f\psi, \cdots, h\chi$ are compositions of numerical functions as defined in § 6. Thus composition of numerical functions is a special case of multiplication (as above defined) of extended numerical functions. The *sum* $F + G$ of F, G is the extended numerical function

$$af + \cdots + ch + p\phi + \cdots + r\chi.$$

Hence, *the set of all extended numerical functions is closed under addition and multiplication as above, and these operations are abstractly identical with addition and multiplication in a field.*

As in § 8 we next define the generator of the F above,

$$\sum_\alpha F(\alpha)w = \sum_\alpha [f(\alpha) + g(\alpha) + \cdots + h(\alpha)]w,$$
$$= \sum_\alpha f(\alpha)w + \sum_\alpha g(\alpha)w + \cdots + \sum_\alpha h(\alpha)w.$$

It follows without difficulty (on comparing coefficients of like M-integers w after distribution of the left of the next equation and rearrangement of the result into a $\sum_\alpha$) that if $\sum_\alpha F(\alpha)w$, $\sum_\alpha H(\alpha)w$ are given generators as just defined of extended numerical functions, there exists a unique generator $\sum_\alpha G(\alpha)w$ such that

$$\sum_\alpha F(\alpha)w \times \sum_\alpha G(\alpha)w = \sum_\alpha H(\alpha)w$$

when and only when $F(\epsilon) \neq 0$. Provided $F(\epsilon) \neq 0$, the extended numerical function whose generator is $\sum_\alpha G(\alpha)w$ is called the *quotient* of that whose generator is $\sum_\alpha H(\alpha)w$ by that whose generator is $\sum_\alpha F(\alpha)w$, and the function generated by $\sum_\alpha F(\alpha)w$ is called *regular* or *irregular* according as $F(\epsilon) \neq 0$ or $F(\epsilon) = 0$.

The set of all extended numerical functions is an irregular field in which the irregular elements are the irregular functions and the four fundamental operations are as above defined.*

11. *Functions of extent* 1 *in the elliptic case.* When all the prime functions occurring in a given application of the preceding theory are of extent 1, the relations between compositions and separations of numerical functions are particularly simple and interesting, as they are abstractly identical with the algebra of the majority of arithmetical functions of rational integers depending upon the unique factorization law that occurs in the literature. Included in this very special case is another, in which the theory is abstractly identical also with the algebra of the elliptic theta functions and their quotients. To illustrate several of the preceding definitions and theorems we shall give a few examples of the simplest relations in the last instance, the elliptic case.

The theorems selected can be most simply stated in terms of the following classes of M-integers. If each of the coordinates a_j $(j = 1, \cdots, k)$ of the M-integer $\alpha = (a_1, \cdots, a_k)$ is represented in the form $\phi \equiv \phi(x, y, \cdots)$, in which $x, y, \cdots$ are indeterminates, we say that α is of *class* ϕ. For example, if each a_j is a square, necessarily of a rational integer, α is of class x^2; if each a_j is a triangular number, α is of class $x(x+1)/2$; if each a_j is a

* For the postulates of such a field see *Annals of Mathematics*, Vol. 27 (1925), p. 512. The only distinction between a regular and an irregular field is the exclusion of an infinity of irregular divisors in the latter instead of but one in the former.

pentagonal number, α is of class $(3x^2 \pm x)/2$; if each a_j is the sum of 2 squares, α is of class $x^2 + y^2$, etc. In each of these examples ϕ is a quadratic form with rational number coefficients. This characterizes the elliptic case above described; the forms may be in any number of indeterminates.

Let α be any M-integer, π any M-prime, and x a real or complex number. A uniform function of x which is also a numerical function of α may be designated by $f(x, \alpha)$. If however we wish to emphasize $f(x, \alpha)$ quâ function of α, or quâ function of x, we shall write $f(x, \alpha)$ in either of the equivalent forms $f_x(\alpha)$, $f_\alpha(x)$, so that

$$f(x, \alpha) \equiv f_x(\alpha) \equiv f_\alpha(x).$$

This device also simplifies printing.

Let $U(x, \alpha)$, $P(x, \alpha)$ be normal functions of α,

$$U(x, \alpha) = U_\alpha(x) = U_x(\alpha), \quad U_x(\epsilon) = 1,$$
$$P(x, \alpha) = P_\alpha(x) = P_x(\alpha), \quad P_x(\epsilon) = 1,$$

defined by the properties: $P_x(\pi^n) = x^n$; $U_x(\pi^n) = 0$ or x according as $n > 1$ or $n = 1$, where n is a rational integer. Then (see § 8) we have the generating identities

$$\Pi_\pi (1 + xz) = \Sigma_\alpha U_x(\alpha) w, \quad \Pi_\pi (1 - xz)^{-1} = \Sigma_\alpha P_x(\alpha) w,$$

and therefore

$$U_x P_{-x} = e, \quad P_x = U_{-x}^{-1}, \quad U_x = P_{-x}^{-1};$$

that is, (see § 6) U_x, P_{-x} are reciprocals and each is a prime function. Referring to § 4, (4)-(6), we see that

$$U_x(\alpha) = \mu^2(\alpha) x^{\gamma(\alpha)}, \quad P_x(\alpha) = x^{\rho(\alpha)},$$

and hence in particular,

$$U_{-1} = \mu, \quad P_{-1} = \lambda, \quad U_1 = (\mu^2), \quad P_1 = 1;$$

the significance of the parenthesis in (μ^2) is explained in § 6, end. Hence

$$\mu P_1 = \lambda(\mu^2) = e,$$

which in non-symbolic form state that each of the sums

$$\Sigma \mu(\tau), \quad \Sigma \lambda(\tau) \mu^2 (\alpha/\tau),$$

where Σ refers to all divisors τ of α, has the value 1 or 0 according as $\alpha = \epsilon$ or $\alpha \neq \epsilon$. These are abstractly identical with the corresponding theorems con-

cerning rational integers for the functions usually denoted by μ, λ. By the theory already developed the like extension to M-integers holds for any of the uniform functions of divisors of rational integers.

The separations U_x', P_x', are now known from the definitions, but they may be defined independently from the generators. The two definitions are of course identical. Consider then

$$\Pi_n (1 + xz^n), \qquad \Pi_n (1 - xz^n)^{-1} \qquad (\text{mod } C),$$

in which the coefficients of z^n are respectively (after distribution of the products)

$$U_x'(\pi^n), \qquad P_x'(\pi^n) \qquad (\text{mod } C).$$

Let $q_i(n)$ be the number of partitions of n into precisely i parts, no two of which are equal, and all of which are in C. Then, $\sum_i$ indicating a sum extending to $i = 1, 2, 3, \cdots$, we have

$$U_x'(\pi^n) = \sum_i q_i(n) x^i.$$

For $n = 1, \cdots, 6$ and C the set of all rational integers > 0, we thus have

$$\begin{aligned} U_x'(\pi) &= U_x'(\pi^2) = x, \\ U_x'(\pi^3) &= U_x'(\pi^4) = x^2 + x, \\ U_x'(\pi^5) &= U_x'(\pi^6) = 2x^2 + x. \end{aligned}$$

The equality of successive pairs is not general. Hereafter, unless otherwise stated, C is any set as before. Similarly, if $\sum_r$ refers to all sets of rational integer solutions $r_1, r_2, r_3, \cdots, \geqq 0$ of

$$n = r_1 n_1 + r_2 n_2 + r_3 n_3 + \cdots,$$

where no two of $n_1, n_2, n_3, \cdots$ are equal and all are in C, we have

$$P_x'(\pi^n) = \sum_r x^{r_1 + r_2 + r_3 + \cdots}.$$

Since U_x, P_x are normal, $U_x'(\alpha)$, $P_x'(\alpha)$ (mod C) are now known by § 4, end, for all M-integers α. Note that π, α play the part of indeterminates, disappearing from the final forms. Thus if

$$\alpha = \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_k^{a_k}$$

is the resolution of α into powers of M-primes,

$$U_x'(\alpha) = \prod_{j=1}^{k} [\textstyle\sum_i q_i(a_j) x^i],$$

and similarly for the other.

Having found these explicit forms we may examine now the meaning in terms of rational integers of $U_x' P_{-x}' = e$. Let α be as above. Then all the divisors δ of α are

$$\delta = \pi_1^{d_1} \pi_2^{d_2} \cdots \pi_k^{d_k}, \quad 0 \leqq d_j \leqq a_j \qquad (j = 1, \cdots, k),$$

and the divisor δ' conjugate to that just written is

$$\delta' = \pi_1^{a_1 - d_1} \pi_2^{a_2 - d_2} \cdots \pi_k^{a_k - d_k}.$$

Form $U_x'(\delta)$, $P_{-x}'(\delta')$ as above for every pair (δ, δ'), take the product of the results and sum it with respect to all pairs (δ, δ'). Then this sum has the value 1 or 0 according as the given α is or is not ϵ or, what is the same, according as $(a_1, \cdots, a_k)$ is or is not $(0, \cdots, 0)$. This obviously is a property of compound partitions alone. The like is true, by the theory developed, for every theorem concerning compositions and separations of numerical functions.

By giving to x special values, and by properly choosing the set C which has been taken as the modulus in constructing $U'(\alpha)$, $P'(\alpha)$, we can apply the known product developments of the elliptic theta functions and constants to a great variety of separations of numerical functions. The following will be sufficient as examples. In addition to the functions λ, μ, ρ already defined we shall need $u(\alpha)$, which has the value 1 for all M-integers α; $\theta(\alpha)$, $=$ the number of decompositions of the M-integer α into a pair of coprime M-integer factors. In the last, if $\alpha = \beta\gamma$ is one such decomposition, and $\beta \neq \gamma$, $\gamma\beta$ is counted as another.

As in § 7, end, we consider several sets: $C_0 =$ the set of all integers > 0; $C_1 =$ the set of all odd integers > 0; $C_2 =$ the set of all even integers > 0. By the notation already explained in § 7, if ϕ is any numerical function, ϕ_j' is the separation ϕ' (mod C_j). Hence λ_j', μ_j', θ_j', $\cdots$ $(j = 0, 1, 2)$ are defined. For clearness we recall the necessary generators,

$$\begin{array}{rcl}
\mu, & \text{generator} & 1 - z; \\
\lambda, & \text{"} & (1 + z)^{-1}; \\
(\mu^2), & \text{"} & (1 + z); \\
\theta, & \text{"} & (1 + z)(1 - z)^{-1}; \\
(\theta\lambda), & \text{"} & (1 - z)(1 + z)^{-1}; \\
\nu, & \text{"} & (1 - z)^{-2}; \\
u, & \text{"} & (1 - z)^{-1};
\end{array}$$

$\nu(\alpha)$, as in § 2, is the number of divisors of α. From these generators we see by inspection numerous compositions. For example $\lambda(\mu^2) = e$, so that λ, (μ^2) are reciprocals; $\theta = u(\mu^2)$; $\nu = u^2$; $\mu\nu = u$; etc.

The following illustrates the way in which the remaining examples are

written down. We have the well-known product expansion from the theta functions,

$$\Pi\,(1-z^j)(1+z^j)^{-1}=1+2\,\Sigma\,(-1)^j z^{j^2}, \qquad (j=1,2,\cdots),$$

and hence the theorem: the separation $(\theta\lambda)_0'$ vanishes for all arguments $\alpha\neq\epsilon$ not in class x^2 (see beginning of this section for definition of class); if the argument is the M-integer $(a_1^2, a_2^2, \cdots, a_k^2)$, where $a_j\geqq 0$ $(j=1, \cdots, k)$ and $a_1+a_2+\cdots+a_k>0$, the separation has the value $2(-1)^{a_1+a_2+\cdots+a_k}$; if the argument is ϵ the value of the separation is 1. The significance of the parentheses in $(\theta\lambda)$ is as in § 6, end. Note that $(\theta\lambda)'$, for any given C, is not equal to $\theta'\lambda'$.

The theta expansions used in writing down the following are all classical and easily recognized; there is no need to transcribe them. Write for the moment $\phi\equiv\mu^3$. Then the separation ϕ_0' has the value zero if its argument $\alpha\neq\epsilon$ is not in class $x\,(3x\pm 1)/2$; if $\alpha=\epsilon$ the value is 1; if

$$\alpha=(a_1(3a_1\pm 1)/2, \cdots, a_k(3a_k\pm 1)/2)$$

the value is $(-1)^{a_1+\cdots+a_k}$. Next, write $\phi\equiv\mu(\mu^2)$. Then ϕ_0' vanishes for all arguments not in class $x(x-1)/2$, while for such arguments it has the value 1. The next illustrates the occurrence of two sets in one relation. Write $\phi\equiv(\mu^2)^2$. Then the separation (which is a composition of two separations, or a separation of a composition of two functions) $\mu_2'\ \phi_1'$ vanishes if its argument $\alpha\neq\epsilon$ is not in class x^2; if the argument is $(a_1^2, \cdots, a_k^2)$, where $a_j\geqq 0$ $(j=1,\cdots,k)$ and $a_1+\cdots+a_k>0$, the value is 2. Such results can be multiplied indefinitely.

The next is one of a large class of similar results. Let $E(n)$ have its usual meaning as a binary quadratic class number function; write $\chi\equiv\mu^3$, $\psi\equiv(\mu^2)^6$. Then the separation $\chi_2'\ \psi_1'$ for the argument $\alpha=(a_1,\cdots,a_k)$ has the value $(12(-1)^{a_1}E(a_1), \cdots, 12(-1)^{a_k}E(a_k))$. To recall the meaning of $\chi\psi$ in this instance, let $\alpha=\alpha_1\alpha_2\cdots\alpha_9$ be any decomposition of α into a product of nine factors. Then the value of $\chi\psi$ for the argument α is

$$\Sigma\,\{\mu(\alpha_1)\mu(\alpha_2)\mu(\alpha_3)\,\mu^2(\alpha_4)\mu^2(\alpha_5)\cdots\mu^2(\alpha_9)\},$$

the sum extending to all sets $\alpha_1, \alpha_2, \cdots, \alpha_9$. These examples, which can be continued indefinitely, are enough to show the meanings of previous definitions and theorems, which is all that is necessary here. If instead of specializing x to be ± 1 in $U_x(\alpha)$, $P_x(\alpha)$ as in these examples for $\lambda, \mu, \cdots$, we take $x=\exp(\pm iy)$, where i is the imaginary unit, and combine results as directed by the product expansions of elliptic and theta functions of y, we

reach theorems concerning compositions of separations, or vice-versa, which involve sines and cosines of integer multiples of y. These results can then be paraphrased, as in previous papers, into separation-composition theorems containing functions wholly arbitrary except as to parity. All theorems such as the above concerning specific functions λ, μ, $\cdots$ are special cases of such general paraphrased separation-compositions.

12. *Addition and subtraction theorems.* Several of the examples in the preceding section illustrate a general situation concerning separations with respect to sets that are either the logical sum or product, or the logical difference of given sets. If the set C contains ($=$ includes) the set C_j, we write $C \mid C_j$, as in the usual generalization of arithmetical division in the theory of ideals. Sets having no element common are called *coprime*; the *sum* $C_i + C_j$ of two sets is their logical sum; their *product* C_iC_j is their logical product. Similarly for any number of sets. If C_j $(j=1, \cdots, s)$ are such that every pair of them are coprime, the s sets are called coprime. The condition that C_j $(j=1, \cdots, s)$ be coprime is $C_1C_2\cdots C_s =$ the null set.

Let C_j $(j=1, \cdots, s)$ be s coprime sets, and write $C = C_1 + C_2 + \cdots + C_s$. Let ϕ' be the separation of ϕ (mod C), and ϕ_j' that of ϕ (mod C_j) $(j=1, \cdots, s)$. Then, from a consideration of the respective generators, it follows at once that

$$\phi' = \phi_1'\phi_2'\cdots\phi_s'; \tag{26}$$

that is, *the separation of any numerical function modulo the sum of s coprime sets is equal to the composition of the separations of s functions, each equal to the given function, taken to the s moduli composing the sum.*

Let $C_a, C_b, \cdots, C_t$ be any distinct $s-r$ sets, $r>0$, chosen from the sets C_j $(j=1, \cdots, s)$, and let the remaining r sets be $C_\alpha, C_\beta, \cdots, C_\tau$, so that in some order the $C_a, \cdots, C_t, C_\alpha, \cdots, C_\tau$ are identical with the $C_1, \cdots, C_s$. Let ϕ' be as before, and let ϕ_h' be the separation (mod C_h) of ϕ $(h=a, b, \cdots, t, \alpha, \beta, \cdots, \tau)$; also write ψ_h for the reciprocal of ϕ_h, and let ψ_h' be the separation of ψ_h. Write $\Gamma = C - C_\alpha - C_\beta - \cdots - C_\tau$; namely, Γ is the set which remains when from C are deleted all elements of $C_\alpha + C_\beta + \cdots + C_\tau$, and let ϕ'' be the separation of ϕ (mod Γ). Then

$$\phi'' = \phi_a'\phi_b'\cdots\phi_c' = \phi'\psi_a'\psi_\beta'\cdots\psi_\gamma'. \tag{27}$$

On the basis of such considerations we can apply the arithmetic of logic developed in a forthcoming paper to construct an arithmetic (including the theory of congruences) of separations with reference to their sets.

Lagrange Resolvents in Euclidean Geometry.

By Leonard M. Blumenthal.

Introduction.

1. *An Invariant and its Geometrical Interpretation.* Consider the combination

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4)$$

of the four points x_1, x_2, x_3, x_4, in the complex plane. Here $\bar{x}_i$ is the conjugate of x_i. It is seen at once that this function is invariant under the translation $y = x + a$, and also under the rotation $y = tx$ where t is a turn, a complex quantity with unit modulus. For

$$(y_1 - y_3)(\bar{y}_2 - \bar{y}_4) = (tx_1 + a - tx_3 - a)(\bar{x}_2/t + a - \bar{x}_4/t - a)$$
$$= (x_1 - x_3)(\bar{x}_2 - \bar{x}_4) \; .$$

The four points x_1, x_2, x_3, x_4 determine an ordered quadrangle. Denoting by δ_{13} and δ_{24} the lengths of the strokes $x_1 - x_3$ and $x_2 - x_4$ respectively, we may write

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) = \delta_{13}\delta_{24}e^{i\theta} = \delta_{13}\delta_{24}\cos\theta + i\,\delta_{13}\delta_{24}\sin\theta,$$

where θ is the angle which $x_1 - x_3$ makes with $x_2 - x_4$. If x_0 is the point of intersection of $x_1 - x_3$ and $x_2 - x_4$,

$$\delta_{13} = \delta_{01} + \delta_{03}, \qquad \delta_{24} = \delta_{02} + \delta_{04},$$

and

$$\delta_{13}\delta_{24}\cos\theta = \tfrac{1}{2}(\delta_{12}{}^2 - \delta_{23}{}^2 + \delta_{34}{}^2 - \delta_{41}{}^2) = 2N.$$

We shall speak of N as the *norm* of the quadrangle.

Also we have

$$\delta_{13}\delta_{24}\sin\theta = 2A,$$

where A denotes the area of the quadrangle. Hence we may write

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) = 2(N + iA)\,;$$

i. e., the invariant we set out to examine is seen to express the area and norm

of the quadrangle determined by the points x_i $(i = 1, 2, 3, 4)$. We shall refer to the expression $2(N + iA)$ as the *norm-area* of the polygon.

If we write $x_1 - x_3 = v$ and $x_2 - x_4 = u$, the invariant takes the form

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) = v\bar{u};$$

and since each of v and u is invariant under translations we have the following

THEOREM. *The norm-area of a quadrangle is unaltered under a translation of either diagonal.*

2. *The Lagrange Resolvent.* In a memoir * devoted to the fundamental principles of the solutions of cubic and quartic equations, Lagrange introduced the resolvent

$$v = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \cdots + \epsilon^{n-1} x_n$$

where ϵ is a complex nth root of unity. By this notation we shall further understand ϵ to be a primitive root.

It may be observed at once that the resolvent v is invariant under translations $y = x + a$, since

$$\epsilon^{n-1} + \epsilon^{n-2} + \cdots + \epsilon + 1 = 0$$

and further, that combinations like $v\bar{u}$ (u being a similar resolvent) are invariant under rotations $y = tx$.

An extension of the example in § 1 suggests itself. We seek to know when it is possible to express the norm-area of an ordered polygon in terms of the Lagrange resolvents (and their conjugates) of polygons into which the given polygon is decomposed. Viewed from another angle, we wish to know "how much" of a polygon may be translated without altering its norm-area.

The investigations presented here prove that the norm-area of a $2n$-gon may be expressed in terms of the Lagrange resolvents of the two n-gons of which it is composd. The coefficients in the expression are given explicitly as very simple functions of n.

It is shown that such an expression in terms of the diagonal strokes and their conjugates is possible only in the case of the quadrangle.

---

* Memoirs of Berlin Academy, 1769; reprinted in *Oeuvres de Lagrange* (Paris, 1868), Vol. 3, p. 207.

It is further proved that the areas of polygons of mn sides ($m \neq 2$) are not expressible in terms of the Lagrange resolvents of the m n-gons. It may be noted that the areas of polygons are obtained without resort to the "triangulation" process; that is, the areas of polygons are obtained as the imaginary part of the expression for the norm-area of the polygons.

It is finally shown that the combination $v_i\bar{u}_i$ is the only invariant combination of the Lagrange resolvents of the polygons that is unaltered by the group of substitutions $(1, 3, 5, \cdots, 2n-1)(2, 4, 6, \cdots, 2n)$. The expression for norm-area may be referred to as a *lineo-linear* invariant of two n-gons.

Part One. Polygons of 2n Sides.

3. *The Hexagon as Two Triangles.* As indicated in Fig. 1, the hexagon is considered as being formed by the vertices of the two triangles $x_1x_3x_5$ and $x_4x_6x_2$.

Now the determinant *

$$\begin{vmatrix} x_1 & x_3 & x_5 \\ \bar{x}_4 & \bar{x}_6 & \bar{x}_2 \\ 1 & 1 & 1 \end{vmatrix}$$

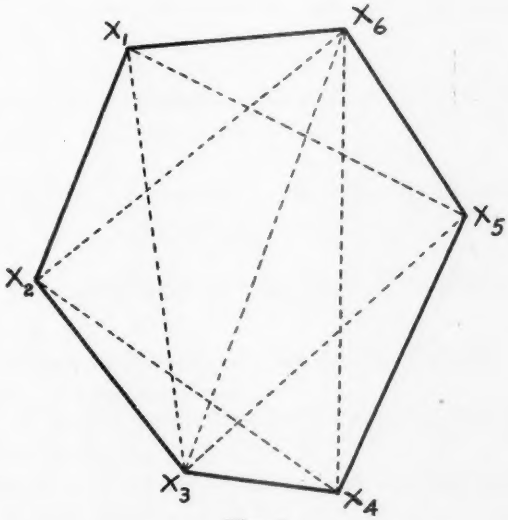

Fig. 1.

* The determinant set equal to zero is the condition that the two triangles be negatively similar.

is seen to be unaltered by the translation $y = x + a$ and also by the rotation $y = tx$. Denoting this determinant by Δ, we have

$$\begin{aligned} -\Delta &= x_1\bar{x}_2 - \bar{x}_2x_3 + x_3\bar{x}_4 - \bar{x}_4x_5 + x_5\bar{x}_6 - \bar{x}_6x_1 \\ &= (x_1 - x_3)(\bar{x}_2 - \bar{x}_6) + (x_3 - x_5)(\bar{x}_4 - \bar{x}_6). \end{aligned}$$

We may conclude from § 1 that

$$\begin{aligned} (x_1 - x_3)(\bar{x}_2 - \bar{x}_6) &= \tfrac{1}{2}(\delta^2_{12} - \delta^2_{23} + \delta^2_{36} - \delta^2_{61}) + 2iA_1 \\ (x_3 - x_5)(\bar{x}_4 - \bar{x}_6) &= \tfrac{1}{2}(\delta^2_{34} - \delta^2_{45} + \delta^2_{56} - \delta^2_{36}) + 2iA_2 \end{aligned}$$

where A_1 and A_2 are respectively the areas of quadrangles $x_1x_2x_3x_6$ and $x_3x_4x_5x_6$. Adding, we obtain

$$-\Delta = 2(N + iA)$$

where N and A are the norm and area, respectively, of the hexagon.

To express Δ in terms of the Lagrange resolvents of the two triangles,

$$\begin{aligned} v_1 &= x_1 + \omega x_3 + \omega^2 x_5; \quad & u_1 &= x_4 + \omega x_6 + \omega^2 x_2 \\ v_2 &= x_1 + \omega^2 x_3 + \omega x_5; \quad & u_2 &= x_4 + \omega^2 x_6 + \omega x_2 \end{aligned}$$

where ω is an imaginary cube root of unity, we note that

$$\Delta = \begin{vmatrix} x_1 + \omega x_3 + \omega^2 x_5, & x_3 - x_5, & x_5 \\ \bar{x}_4 + \omega\bar{x}_6 + \omega^2\bar{x}_2, & \bar{x}_6 - \bar{x}_2, & \bar{x}_2 \\ 0 & , \ 0 & , \ 1 \end{vmatrix};$$

whence, multiplying the second column by $-\omega(1-\omega)$ and adding to it the first column yields

$$\Delta = -[1/\omega(1-\omega)]\,(v_1\bar{u}_1 - v_2\bar{u}_2)$$

and hence *

$$[1/\omega(1-\omega)]\,(v_1\bar{u}_1 - v_2\bar{u}_2) = 2(N + iA).$$

Since the resolvents are invariant under translations, we have proved the

---

* Looking towards the later developpment of the paper we note here that, for

$$u_1 = x_2 + \omega x_4 + \omega^2 x_6 \quad \text{and} \quad u_2 = x_2 + \omega^2 x_4 + \omega x_6,$$

we have

$$[1/(1-\omega)]\,(v_1\bar{u}_1 - \omega v_2\bar{u}_2) = 2(N + iA),$$

which may be obtained from the expression above by substituting $\omega\bar{u}_1$ for $\bar{u}_1$ and $\omega^2\bar{u}_2$ for $\bar{u}_2$.

THEOREM. *The norm-area of a hexagon is unaltered by translating either of the component triangles.*

4. *The Decagon.* As the analysis to be employed in the case of the $2n$-gon can be well illustrated in the treatment of the decagon, we shall ex-

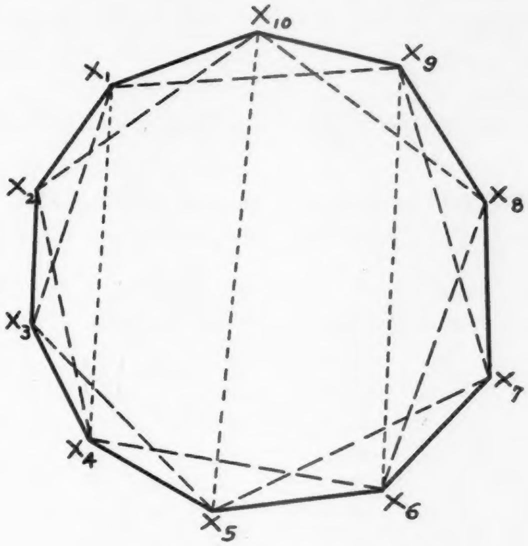

Fig. 2.

amine this case in some detail. Splitting the decagon into the four quadrangles indicated in the figure, we obtain

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) + (x_1 - x_5)(\bar{x}_4 - \bar{x}_{10}) + (x_5 - x_9)(\bar{x}_6 - \bar{x}_{10}) + (x_9 - x_7)(\bar{x}_6 - \bar{x}_8) = \sum_{k=1}^{5} \bar{x}_{2k}(x_{2k-1} - x_{2k+1}) = 2(N + iA)$$

where N and A are the norm and area, respectively, of the decagon. The Lagrange resolvents considered are

$$\begin{aligned}
v_1 &= x_1 + \epsilon x_3 + \epsilon^2 x_5 + \epsilon^3 x_7 + \epsilon^4 x_9; & u_1 &= x_2 + \epsilon x_4 + \epsilon^2 x_6 + \epsilon^3 x_8 + \epsilon^4 x_{10}\\
v_2 &= x_1 + \epsilon^2 x_3 + \epsilon^4 x_5 + \epsilon x_7 + \epsilon^3 x_9; & u_2 &= x_2 + \epsilon^2 x_4 + \epsilon^4 x_6 + \epsilon x_8 + \epsilon^3 x_{10}\\
v_3 &= x_1 + \epsilon^3 x_3 + \epsilon x_5 + \epsilon^4 x_7 + \epsilon^2 x_9; & u_3 &= x_2 + \epsilon^3 x_4 + \epsilon x_6 + \epsilon^4 x_8 + \epsilon^2 x_{10}\\
v_4 &= x_1 + \epsilon^4 x_3 + \epsilon^3 x_5 + \epsilon^2 x_7 + \epsilon x_9; & u_4 &= x_2 + \epsilon^4 x_4 + \epsilon^3 x_6 + \epsilon^2 x_8 + \epsilon x_{10}
\end{aligned}$$

where ϵ is a primitive fifth root of unity.

The products $v_i\bar{u}_i$ $(i=1, 2, 3, 4)$ can be best written in the matrix from

$$v_1\bar{u}_1 = \begin{array}{c|ccccc} & x_1 & x_3 & x_5 & x_7 & x_9 \\ \hline \bar{x}_2 & 1 & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 \\ \bar{x}_4 & \epsilon^4 & 1 & \epsilon & \epsilon^2 & \epsilon^3 \\ \bar{x}_6 & \epsilon^3 & \epsilon^4 & 1 & \epsilon & \epsilon^2 \\ \bar{x}_8 & \epsilon^2 & \epsilon^3 & \epsilon^4 & 1 & \epsilon \\ \bar{x}_{10} & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 & 1 \end{array}$$

Noting that the determinant of the matrix is a circulant,* we write symbolically,

$$v_1\bar{u}_1 = C(1\,\epsilon\,\epsilon^2\,\epsilon^3\,\epsilon^4)\,; \qquad v_3\bar{u}_3 = C(1\,\epsilon^3\,\epsilon\,\epsilon^4\,\epsilon^2)\,;$$
$$v_2\bar{u}_2 = C(1\,\epsilon^2\,\epsilon^4\,\epsilon\,\epsilon^3)\,; \qquad v_4\bar{u}_4 = C(1\,\epsilon^4\,\epsilon^3\,\epsilon^2\,\epsilon).$$

Writing the function of the vertices $x_1, x_2, \cdots, x_{10}$ obtained as representing the norm-area of the decagon, in matrix form

$$\text{(I)} \qquad \begin{array}{c|ccccc} & x_1 & x_3 & x_5 & x_7 & x_9 \\ \hline \bar{x}_2 & 1 & -1 & 0 & 0 & 0 \\ \bar{x}_4 & 0 & 1 & -1 & 0 & 0 \\ \bar{x}_6 & 0 & 0 & 1 & -1 & 0 \\ \bar{x}_8 & 0 & 0 & 0 & 1 & -1 \\ \bar{x}_{10} & -1 & 0 & 0 & 0 & 1 \end{array}$$

it is clear that our problem resolves itself into finding multipliers A_1, A_2, A_3, A_4 such that the matrix

$$A_1v_1\bar{u}_1 + A_2v_2\bar{u}_2 + A_3v_3\bar{u}_3 + A_4v_4\bar{u}_4$$

will be identical with the matrix (I). Equating coefficients leads to only four independent equations:

$$A_1 + A_2 + A_3 + A_4 = 1,$$
$$\epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3 + \epsilon^4 A_4 = -1,$$
$$\epsilon^2 A_1 + \epsilon^4 A_2 + \epsilon A_3 + \epsilon^3 A_4 = 0,$$
$$\epsilon^3 A_1 + \epsilon A_2 + \epsilon^4 A_3 + \epsilon^2 A_4 = 0;$$

---

* We should expect the products $v_i\bar{u}_i$ to be of this form, since, it is well-known that if $1, i_1, i_2, \cdots, i_n$ be the $n+1$ roots of $x^{n+1}-1=0$, then for any set $A, A_1, \cdots, A_n$ we have (Spottiswoode, *Crelle's Journal für Mathematik,* Vol. 51, pp. 209-271)

$$\begin{vmatrix} A & A_1 & A_2 \cdots & A_n \\ A_n & A & A_1 \cdots & A_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ A_1 & A_2 & A_3 \cdots & A \end{vmatrix} = (-1)^{n(n-1)/2}(A + A_1 + \cdots + A_n)$$
$$(A + i_1A_1 + i_1^2A_2 + \cdots + i_1^nA_n)$$
$$\cdots (A + i_nA_1 + i_n^2A_2 + \cdots + i_n^nA_n).$$

whence

$$5A_1 = 1 - \epsilon^4, \quad 5A_2 = 1 - \epsilon^3, \quad 5A_3 = 1 - \epsilon^2, \quad 5A_4 = 1 - \epsilon.$$

Hence

$$(1/5)[(1-\epsilon^4)v_1\bar{u}_1 + (1-\epsilon^3)v_2\bar{u}_2 + (1-\epsilon^2)v_3\bar{u}_3 + (1-\epsilon)v_4\bar{u}_4] = 2(N + iA)$$

is the desired expression for the norm-area of the decagon in terms of the Lagrange resolvents of the two pentagons whose vertices form the decagon. Since the resolvents are invariants under translations, we have proved the

THEOREM. *The norm-area of a decagon is unaltered by translating either of its component pentagons.*

5. *General Case. The $2n$-gon.* We write the resolvents of each n-gon in the compact form

$$v_j = \sum_{k=0}^{n-1} \epsilon^{jk} x_{2k+1} \qquad (j = 1, 2, \cdots, n-1)$$

$$\bar{u}_j = \sum_{k=0}^{n-1} \epsilon^{j(n-k)} \bar{x}_{2(k+1)} \qquad (j = 1, 2, \cdots, n-1)$$

Now the $2n$-gon may be divided into $(n-1)$ quadrangles whose vertices are given in the rows of

1	2	3	4
$2n$	1	4	5
$2n-1$	$2n$	5	6
$2n-2$	$2n-1$	6	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

and the areas of which exhaust the area of the $2n$-gon. We have

$$x_1\bar{x}_2 - \bar{x}_2 x_3 + x_3\bar{x}_4 - \cdots + x_{2n-1}\bar{x}_{2n} - \bar{x}_{2n}x_1 =$$

$$\sum_{k=1}^{n} \bar{x}_{2k}(x_{2k-1} - x_{2k+1}) = 2(N + iA).$$

In matrix form, this is

$$\text{(II)} \qquad \begin{array}{c|cccccccc}
 & x_1 & x_3 & x_5 & x_7 & \cdots & x_{2n-3} & x_{2n-1} \\
\hline
\bar{x}_2 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
\bar{x}_4 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
\bar{x}_6 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
\bar{x}_8 & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\bar{x}_{2n-2} & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\
\bar{x}_{2n} & -1 & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}$$

The products $v_i\bar{u}_i$ we write, symbolically, as

$$v_1\bar{u}_1 = C(1\ \epsilon\ \epsilon^2 \cdots \epsilon^{n-1})$$
$$v_2\bar{u}_2 = C(1\ \epsilon^2\ \epsilon^4 \cdots \epsilon^{(2n-1)})$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$v_{n-1}\bar{u}_{n-1} = C(1\ \epsilon^{n-1}\ \epsilon^{2(n-1)} \cdots \epsilon^{(n-1)(n-1)})$$

Again our problem is to find multipliers A_i $(i = 1, 2, \cdots, n-1)$ so that the matrix

$$\sum_{i=1}^{n-1} A_i v_i \bar{u}_i$$

will be identical with the matrix (II).

We note that ϵ being a primitive root of $x^n - 1 = 0$ corresponding elements of corresponding rows of the matrices $v_i\bar{u}_i$ are

$$1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$$

in some order. The desired multipliers A_i are determined by the $n-1$ linear, non-homogeneous equations;

$$\epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3 + \cdots + \epsilon^{n-1} A_{n-1} = -1$$
$$\epsilon^2 A_1 + \epsilon^4 A_2 + \epsilon^6 A_3 + \cdots + \epsilon^{2(n-1)} A_{n-1} = 0$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\epsilon^{n-1} A_1 + \epsilon^{2(n-1)} A_2 + \epsilon^{3(n-1)} A_3 + \cdots \epsilon^{(n-1)(n-1)} A_{n-1} = 0.$$

These equations possess a unique solution, for the determinant of the coefficients may be shown to be different from zero (indeed, evaluated) in the following manner.*

$$\begin{vmatrix} \epsilon & \epsilon^2 & \epsilon^3 \cdots & \epsilon^{n-1} \\ \epsilon^2 & \epsilon^4 & \epsilon^6 \cdots & \epsilon^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ \epsilon^{n-1} & \epsilon^{2(n-1)} & \cdots & \epsilon^{(n-1)(n-1)} \end{vmatrix} = \epsilon\,\epsilon^2\,\epsilon^3 \cdots \epsilon^{n-1}\,\zeta^{1/2}(\epsilon, \epsilon^2, \cdots, \epsilon^{n-1})$$

the symbol on the right hand side being Sylvester's notation for a continued product of the differences $(\epsilon - \epsilon^2)(\epsilon - \epsilon^3)\cdots$. The function partakes of the nature of a square root, having two values corresponding to various permutations of $\epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$.

Since

$$\epsilon\,\epsilon^2\,\epsilon^3 \cdots \epsilon^{n-1} = \epsilon^{n(n-1)/2} = 1$$

---

* *American Mathematical Monthly*, Vol. 32 (1925), p. 522.

we have the determinant of the coefficients equal to the zeta-ic function.

The square of this function being the discriminant of the equation

$$(1) \qquad x^{n-1} + x^{n-2} + \cdots + x + 1 = 0$$

we have

$$\zeta(\epsilon, \epsilon^2, \cdots, \epsilon^{n-1}) = \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-2} \\ s_1 & s_2 & s_3 & \cdots & s_{n-1} \\ s_2 & s_3 & s_4 & \cdots & s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-2} & s_{n-1} & s_n & \cdots & s_{2(n-2)} \end{vmatrix}$$

where $s_i =$ sum of the ith powers of the roots of (1).

Since $\epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$ are roots of (1)

$$s_0 = n-1,\ s_1 = s_2 = \cdots = s_{n-1} = -1,\ s_n = n-1,\ s_{n+1} = -1, \cdots.$$

Hence

$$\zeta(\epsilon, \epsilon^2, \cdots, \epsilon^{n-1}) = \begin{vmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & n-1 \\ -1 & -1 & -1 & \cdots & n-1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & n-1 & \cdots & -1 & -1 \end{vmatrix}$$

This determinant can be evaluated readily, yielding

$$\zeta(\epsilon, \epsilon^2, \cdots, \epsilon^{n-1}) = (-1)^{(n-1)(n-2)/2} n^{n-2}.$$

We proceed to solve the set of $n-1$ equations in the A's.

Evidently, the product of A_i by $\zeta^{1/2}(\epsilon, \epsilon^2, \cdots, \epsilon^{n-1})$ is equal to the original determinant where the elements of the ith column have been replaced by $-1, 0, 0, \cdots, 0$, or, after some reductions

$$A_i = (1-\epsilon^{n-i}) / \prod_{i=1}^{n-1} (1-\epsilon^i).$$

But, since ϵ is a primitive root of $x^n - 1 = 0$, we have

$$(x-\epsilon)(x-\epsilon^2)(x-\epsilon^3) \cdots (x-\epsilon^{n-1}) \equiv x^{n-1} + x^{n-2} + \cdots + x + 1.$$

Letting $x = 1$, we obtain

$$(1-\epsilon)(1-\epsilon^2) \cdots (1-\epsilon^{n-1}) = n.$$

Hence

$$A_i = (1/n)(1-\epsilon^{n-i}) \qquad (i = 1, 2, \cdots, n-1)$$

are the solutions of the sets of equations, and therefore are the multipliers we set out to find. Whence the

THEOREM. *The norm-area of a $2n$-gon expressed in terms of the Lagrange resolvents of the two component n-gons is given by*

$$(1/n)\sum_{i=1}^{n-1}(1-\epsilon^{n-i})v_i\bar{u}_i.$$

Since the resolvents are invariant under translations it follows immediately that

THEOREM. *The norm-area of a $2n$-gon is unaltered by translating either of the component n-gons.*

Part Two. Strokes.

6. In the previous sections a $2n$-gon was considered as being composed of two n-gons, and expressions were found giving the norm-area in terms of the Lagrange resolvents of each n-gon.

In this part, the $2n$-gon is thought of as being composed of n 2-gons (i. e., n strokes), and we are concerned in ascertaining whether or not the norm-area of the $2n$-gon can be expressed in terms of the n strokes and their conjugates. As before, this is the problem of determining if the norm-area of a $2n$-gon is invariant under a translation of one of its component strokes.

We have already seen in § 1 that the norm-area of a quadrangle is expressible in terms of its diagonal strokes. We take up the case of the hexagon.

We have from § 3

$$x_1\bar{x}_2-\bar{x}_2x_3+x_3\bar{x}_4-\bar{x}_4x_5+x_5\bar{x}_6-\bar{x}_6x_1=2(N+iA). \tag{1}$$

Let

$$u_1=x_1-x_4;\quad u_2=x_2-x_5;\quad u_3=x_3-x_6$$
$$U_1=x_1+x_4;\quad U_2=x_2+x_5;\quad U_3=x_3+x_6$$

whence

$$x_1=\tfrac{1}{2}(u_1+U_1),\quad x_2=\tfrac{1}{2}(u_2+U_2),\quad x_3=\tfrac{1}{2}(u_3+U_3)$$
$$x_4=\tfrac{1}{2}(U_1-u_1),\quad x_5=\tfrac{1}{2}(U_2-u_2),\quad x_6=\tfrac{1}{2}(U_3-u_3)$$

and substitution in (1) yields *

$$\begin{aligned}(u_2-u_3+U_2-U_3)(\bar{u}_2+\bar{u}_1-\bar{U}_2+\bar{U}_1)&\\ +(u_2+u_1+U_2-U_1)(\bar{u}_3-\bar{u}_2+\bar{U}_2-\bar{U}_3)&\\ =2(N+iA).&\end{aligned}$$

---

* Note that though none of U_1, U_2, U_3 is invariant under translations, the combinations U_2-U_3, U_2-U_1 etc. in which they enter the equation *are* invariant. Hence, of course, N and A are unchanged by translating or rotating the entire hexagon.

The terms containing U_1, U_2, U_3 will not vanish upon expanding the left-hand side of the last equation and hence we have the following

THEOREM. *The norm-area of a hexagon is not expressible in terms of its three diagonal strokes.*

COROLLARY. *The norm-area of a hexagon is not invariant under a translation of one of its diagonal strokes.*

Consider next the case of the octagon. Letting

$$u_1 = x_1 - x_5, \quad u_2 = x_2 - x_6, \quad u_3 = x_3 - x_7, \quad u_4 = x_4 - x_8$$
$$U_1 = x_1 + x_5, \quad U_2 = x_2 + x_6, \quad U_3 = x_3 + x_7, \quad U_4 = x_4 + x_8$$

we get

$$x_1 = \tfrac{1}{2}(u_1 + U_1), \quad x_2 = \tfrac{1}{2}(u_2 + U_2), \quad x_3 = \tfrac{1}{2}(u_3 + U_3)$$
$$x_4 = \tfrac{1}{2}(u_4 + U_4), \quad x_5 = \tfrac{1}{2}(U_1 - u_1), \quad x_6 = \tfrac{1}{2}(U_2 - u_2)$$
$$x_7 = \tfrac{1}{2}(U_3 - u_3), \quad x_8 = \tfrac{1}{2}(U_4 - u_4).$$

Substituting these values in

$$(x_1 - x_3)(\overline{x}_2 - \overline{x}_4) + (x_1 - x_5)\ (\overline{x}_4 - \overline{x}_8) + (x_5 - x_7)\ (\overline{x}_6 - \overline{x}_8) = 2(N + iA)$$

we obtain

$$(U_1 - U_3)(\bar{U}_2 - \bar{U}_4) + (u_1 - u_3)(\bar{u}_2 - \bar{u}_4) + 2u_1\bar{u}_4 = 4(N + iA).$$

As in the case of the hexagon, the terms in U_1, U_2, U_3, U_4 are present and hence the norm-area of the octagon is not invariant under a translation of a single diagonal stroke.

Note, however, that if the same translations are applied to the two strokes $x_1 - x_5$ and $x_3 - x_7$, the rest of the figure being left unaltered, the norm-area of the octagon *is* invariant (since the difference $U_1 - U_3$ is invariant). The same statement may be made with regard to the strokes $x_2 - x_6$ and $x_4 - x_8$.

For the case of the $2n$-gon, we give the formula

$$\sum_{k=1}^{n} (\bar{u}_{2k} + \bar{U}_{2k})\ [(u_{2k-1} + U_{2k-1}) - (u_{2k+1} + U_{2k+1})] = 2(N + iA)$$

where $u_{n+i} \equiv - u_i$; $U_{n+i} \equiv U_i$ $(i = 1, 2, \cdots, n)$ and all subscripts are first to be reduced (mod $2n$), and hence conclude that only in the case of the quadrangle is the norm-area expressible in terms of the diagonal strokes alone.

7. *The mn-gon.* We give the general formula for the area of the mn-gon

$$\sum_{i=0}^{n-1} (\delta_{i,j} v_{i,j} \bar{v}_{i,j+1} - \delta_{i,j} \bar{v}_{i,j} v_{i,j+1}) = 4niA \qquad (j = 1, 2, \cdots, m)$$

where $\delta_{im} \equiv \epsilon^{i}$ $(i = 1, 2, \cdots, n-1)$; $\delta_{i,j} \equiv 1$ $(j \neq m)$ and

$$v_{ij} = x_j + \epsilon^{i} x_{m+j} + \epsilon^{2i} x_{2m+j} + \cdots + \epsilon^{i(n-1)} x_{(n-1)m+j}$$

and conclude that only in the case of the $2n$-gon is the area expressible in terms of the Lagrange resolvents of the m n-gons. Hence, the area of an mn-gon is not invariant under a translation on one of its component n-gons for $m \neq 2$.

8. We have seen that

$$\text{(I)} \qquad (1/n) \sum_{i=1}^{n-1} (1 - \epsilon^{n-i}) v_i \bar{u}_i = 2(N + iA)$$

is an invariant under translations and rotations of the two polygons with vertices $x_1, x_3, \cdots, x_{2n-1}$ and $x_2, x_4, \cdots, x_{2n}$. It is easily seen that (I) is also invariant under the substitutions $(1\,3\,5 \cdots 2n-1)$ $(2\,4\,6 \cdots 2n)$ applied simultaneously to each polygon.

Moreover, the combination $v_i \bar{u}_i$ is the *only* invariant combination of the Lagrange resolvents of the polygon that is unaltered by the above substitutions for

$$v_i \bar{u}_j = (x_1 + \epsilon^{i} x_3 + \epsilon^{2i} x_5 + \cdots + \epsilon^{(n-1)i} x_{2n-1}) (\bar{x}_2 + \epsilon^{j(n-1)} \bar{x}_4 + \cdots + \epsilon^{j} \bar{x}_{2n}).$$

Apply the substitution and denote by $V_i \bar{U}_j$ the transformed $v_i \bar{u}_j$.

We have

$$\begin{aligned} V_i \bar{U}_j &= (x_3 + \epsilon^{i} x_5 + \epsilon^{2i} x_7 + \cdots + \epsilon^{(n-1)i} x_1) (\bar{x}_4 + \epsilon^{j(n-1)} \bar{x}_6 + \cdots + \epsilon^{j} \bar{x}_2) \\ &= \epsilon^{(n-1)i} (x_1 + \epsilon^{i} x_3 + \epsilon^{2i} x_5 + \cdots + \epsilon^{(n-1)i} x_{2n-1}) \cdot \epsilon^{j} (\bar{x}_2 + \epsilon^{j(nn)} x_n + \cdots + \epsilon^{j} \bar{x} \\ &= \epsilon^{j-i} v_i \bar{u}_j \end{aligned}$$

and thus we see that only for $j = i$ is the combination absolutely invariant.

Another invariant of the two polygons, *independent of* (I) *is*

$$\sum_{i=1}^{n-1} v_i \bar{u}_i = \begin{array}{c|ccccc} & x_1 & x_3 & x_5 & \cdots & x_{2n-1} \\ \hline \bar{x}_2 & n-1 & -1 & -1 & \cdots & -1 \\ \bar{x}_4 & -1 & n-1 & -1 & \cdots & -1 \\ \vdots & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \bar{x}_{2n} & -1 & -1 & -1 & \cdots & n-1 \end{array}$$

The Ovals of the Plane Sextic Curve.

By J. H. McDonald.

The maximum number of ovals that a plane sextic can have is eleven. The statement was made without proof by Hilbert that the ovals cannot be external to each other. The proof of this theorem by Wright (*American Journal of Mathematics,* Vol. 29, 1907, p. 305) is complicated by the introduction of considerations which are unnecessary; in particular the discussions relating to the shrinkage of an oval and to the approach of the variable sextic and cubic may be omitted or replaced by a simpler treatment. There is a lack of rigor throughout; the existence of the limit curve with eleven acnodes on which the conclusion depends is not clearly demonstrated. It seems desirable to complete the proof of this theorem which is of some interest not only in itself but on account of more general questions which it suggests. An example of this is given at the end of this communication in the form of a theorem regarding the situation of the intersections of two nodal cubics.

It may be taken as proved that if a sextic with eleven ovals external to each other exists, an infinite number of sextics may be found having ten acnodes and an oval. Let these sextics have equations $S_n = 0$ and let them be designated by S_n. From each S_n a curve $S_n + \kappa C_1{}^n C_2{}^n = 0$ is formed as follows. Let A^n, B^n be two of the acnodes of S_n and $C_1{}^n$, $C_2{}^n$ be cubics through all the acnodes except B^n, A^n respectively. The signs are taken so that $C_1{}^n C_2{}^n > 0$ for points on the oval of S_n. Then for small positive values of κ, $S_n + \kappa C_1{}^n C_2{}^n = 0$ is a sextic having eight acnodes, two small ovals near A^n and B^n lying in the part of the plane where $C_1{}^n C_2{}^n < 0$, and an oval lying within the oval of S_n. The process of reduction starts again with this curve and leads to S_{n+1}, and for sufficiently small values of k the ten acnodes of S_{n+1} are the original eight and two others lying within the ovals arising from A^n and B^n. The last oval is contaisnually diminished and it may first be proved that as n increases the cubics $C_1{}^n$, $C_2{}^n$, cannot approach this oval. The cubics $C_1{}^{n+1}$, $C_2{}^{n+1}$ pass through the nine points of intersection of $C_1{}^n$, $C_2{}^n$ since they belong to the pencil of cubics through the eight fixed acnodes of the sextic and they also pass through the points A^{n+1}, B^{n+1} to which the ovals derived from A^n, B^n have shrunk. But these lie in the region

where $C_1^nC_2^n < 0$ so that if $C_1^{n+1} = \lambda C_1^n + \mu C_2^n$ then λ and μ have the same sign and the entire course of $C_1^{n+1} = 0$ is where $C_1^nC_2^n < 0$ and the argument may be repeated for $C_2^{n+1} = 0$. Since the ovals of S_n, S_{n+1} lie in the region where $C_1^nC_2^n > 0$ the shortest distance from C_1^{n+1} or C_2^{n+1} to the ovals is greater than the shortest distance from C_1^n, C_2^n to the ovals.

The possibility must be considered that in the course of the reduction there may be a confluence of the acnodes of $S_n = 0$. Suppose first that the two variable acnodes converge to a point which is not one of the fixed acnodes. This can only occur if they are accessible to each other without crossing C_1^n, C_2^n. This convergence can be avoided as follows. Denote the variable acnodes by A and B and the fixed acnodes by $12\ldots8$, and determine two cubics by the points $12\ldots7AB$ and $12\ldots7A8$, and a second pair of cubics by the points $12\ldots7AB$ and $12\ldots7B8$, the pairs having a cubic in common. Then if the ovals arising from B and 8 by the use of the first pair of cubics are accessible to each other without crossing the cubics, those arising from A and 8 when the second pair is used are inaccessible to each other without crossing the pair of cubics used. This will certainly be true if the points A, B are sufficiently close which may be assumed since they are supposed to converge. This arrangement being adopted the convergence of A and B can henceforth, if it occurs, only be to one of the fixed acnodes. Suppose it to occur and denote the fixed acnode by 9. Then a cubic through six of the seven acnodes which are remote from the convergent set and through $A9B$ cannot have a double point at A, 9 or B, for that cubic would intersect the sextic in twenty points and so must constitute a part of it. This is impossible since the sextic consists of ten acnodes and an oval. The three convergent acnodes converge in such a way that the cubics through the acnodes $12\ldots7$ and through any two of A, 9, B must in these points make angles with each other approaching zero. A pair of these can always be taken so that the ovals arising from the points omitted in the determination and called A^n and B^n originally lie on opposite sides of the cubics.

To complete the proof it is necessary to arrive at a sextic having eleven acnodes. Let S_n, C_1^n, C_2^n be normalized so that the sum of the squares of the coefficients of each curve equals unity. Then for any curve $S_n + \kappa C_1^nC_2^n = 0$ there is a range of positive values of k such that the further reduction may be made. Let k_n be the upper limit of the values of k for which further reduction is possible, or if k has no upper limit let k_n denote a fixed positive number. With the increase of n the quantity k_n may approach zero. This might be the consequence of fulfilling any of the three following conditions:

first the double points which remain fixed must not cease to be acnodes, second the ovals resulting from A and B must not coalesce, third the last oval must not shrink to a point. In case of failure to fulfill any of these conditions the process of reduction cannot be continued. Taking the conditions in order; if an acnode became a crunode for values of k_n which tend to zero there must be a limiting form of the sextic with ten acnodes and an oval such that when the axes are properly placed the equation becomes $ay^2 + (\ )_3 + \cdots = 0$ where the terms not written are of the order indicated. The line $y = 0$ meets the curve at the origin in three points unless the group $(\ )_3$ vanishes with y. But since the curve has no odd branch there must be an even number of real intersections distinct from the origin, hence the group $(\ )_3$ contains y as a factor. But a unicursal curve of the form $y^2 + y(\ )_2 + \cdots = 0$ has two real branches passing through the origin, as may be seen from consideration of the parametric representation of the curve; hence the limiting curve of the form supposed is impossible, and therefore the value of k_n cannot tend to zero unless the second or third condition is unfulfilled. If the second condition is fulfilled but the value of k_n tends to zero the points A^n, B^n must approach and by the rearrangement of points described above this may be prevented. If the first and second conditions are fulfilled and k_n tends to zero on account of the third condition then the limit form of the curve has eleven acnodes. If all three conditions can be fulfilled it follows that $\kappa_n > g > 0$. But the assumptions that $\kappa_n > g > 0$ and that the last oval is not reduced are contradictory. For letting S be a limit curve of the set S_n the curve S will have an oval within the ovals of S_n and arbitrarily close to them for n sufficiently great, and the curve $S_n + \kappa C_1{}^n C_2{}^n$ can be made to pass through a point within the oval of S withotu requiring $k > g$; hence the limit curve S cannot have an oval. Therefore all three conditions cannot continue to be ful,lled; and since the first two can be it follows that the limit of $k_n = 0$ and S the limit curve has eleven acnodes, which was to be proved.

It might happen that the reduction was prevented by the disappearance of the last oval before those due to A^n and B^n. Supposing this to occur let the remaining oval be regarded as the last one in place of the one previously so considered. Then there will be a limit curve with an oval whose greatest chord is less than that of any other. But this is impossible since the process always replaces the last oval by one of smaller greatest chord. Hence no sextic with eleven ovals external to each other can exist.

Throughout this discussion the existence of limit curves is assumed. There must be at least one limit curve, since if the coefficients of S_n are

$a_1^{(n)} \cdots a_{28}^{(n)}$ and $(a_1^{(n)})^2 + \cdots + (a_{28}^{(n)})^2 = 1$ the set of values $a_1^{(n)} \cdots a_{28}^{(n)}$ must have a condensation. This gives the coefficients of the limit curve S.

The theorem which has been proved indicates limitations on the position of the intersections of two nodal cubics. Two nodal cubics $f_1 = 0$, $f_2 = 0$ cannot intersect in such a way that the curve $f_1 f_2 + c = 0$ will have eleven ovals external to each other. Suppose one cubic to be $y^2 = x^2(x+1)$ and a second to have its loop situated on the left of the first curve and above the loop of the first. Also suppose that the intersections of the cubics are on the branch of the first for which $y > 0$; then there cannot be nine of these intersections because the sextic $f_1 f_2 + c = 0$, $c > 0$, where $f_1 > 0$, $f_2 > 0$ on the left, would have eleven ovals external to each other. For there would be an oval within each loop, one in the triangular space to the right of the node of the second cubic, eight ovals between segments of the cubics and an infinite branch. This curve if it existed could be projected into a finite form with eleven ovals. Such a curve does not exist. Therefore the position of intersections of the cubics which had been assumed is impossible.

On Isometric Systems of Curves and Surfaces.

By C. E. Weatherburn.

In the first part of this paper some new theorems are proved concerning isometric orthogonal systems of curves drawn on any surface. The differential invariants employed in this connection are the two-parametric invariants introduced and discussed by the author in a recent paper entitled "On Differential Invariants in Geometry of Surfaces, with some Applications to Mathematical Physics." * In the second part some analogous properties of isometric systems of surfaces are examined, and an isometric congruence of curves is defined. The differential invariants in this portion of the paper are the ordinary invariants frequently used in applied mathematics.

Isometric Orthogonal Systems of Curves.

1. Consider first an isometric orthogonal system of curves on a given surface. These may be taken as parametric curves $u = \text{const.}$, $v = \text{const.}$ Let $\boldsymbol{r}$ be the position vector of the current point on the surface, and let suffixes 1, 2 denote partial differentiation with respect to u, v respectively. Then, with the usual notation,

$$E = (\partial \boldsymbol{r}/\partial u)^2 = \boldsymbol{r}_1^2; \qquad G = (\partial \boldsymbol{r}/\partial v)^2 = \boldsymbol{r}_2^2.$$

Now the necessary and sufficient condition that the parametric curves may be isometric is

$$(1) \qquad \partial^2/\partial u \partial v \log E/G = 0.$$

We may write this as

$$\partial/\partial u(E_2/E) - \partial/\partial v(G_1/G) = 0$$

from which it follows that †

$$\operatorname{div}\left(\frac{E_2 \boldsymbol{r}_1}{E(EG)^{1/2}} - \frac{G_1 \boldsymbol{r}_2}{G(EG)^{1/2}}\right) = 0,$$

and therefore, if $\boldsymbol{a}$, $\boldsymbol{b}$ are the unit tangents to the parametric curves,

$$(2) \qquad \operatorname{div}(\boldsymbol{a} \operatorname{div} \boldsymbol{b} - \boldsymbol{b} \operatorname{div} \boldsymbol{a}) = 0.$$

* *Quarterly Journal of Pure and Applied Mathematics*, Vol. 50, pp. 230-269. (Cambridge, 1925).

† *Loc. cit.*, § 4.

This equation may be transformed in different ways. First it may be expressed

$$(3) \qquad \boldsymbol{a} \cdot \nabla \operatorname{div} \boldsymbol{b} - \boldsymbol{b} \cdot \nabla \operatorname{div} \boldsymbol{a} = 0.$$

Now $-\operatorname{div} \boldsymbol{b}$ is the geodesic curvature * of the curve $v = \text{const.}$, and div $\boldsymbol{a}$ is that of the curve $u = \text{const.}$ Also the above analysis is reversible; so that if (3) is satisfied the parametric curves form an isometric system. Hence the theorem:

A necessary and sufficient condition that an orthogonal system of curves on a surface may be isometric is that, at any point, the sum of the derivatives of the geodesic curvatures of the curves, each in its own direction, be zero.

Again, the equation (2) may also be expressed

$$\operatorname{div} \{(\boldsymbol{a} \operatorname{div} \boldsymbol{a} + \boldsymbol{b} \operatorname{div} \boldsymbol{b}) \times \boldsymbol{n}\} = 0$$

where $\boldsymbol{n}$ is the unit normal to the surface. Writing

$$\boldsymbol{h} = -(\boldsymbol{a} \operatorname{div} \boldsymbol{a} + \boldsymbol{b} \operatorname{div} \boldsymbol{b})$$

and remembering that curl $\boldsymbol{n} = 0$, we deduce that

$$(4) \qquad \boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{h} = 0.$$

Now we have shown elsewhere † that $\boldsymbol{h}$ is the tangential component of the vector curvature of the orthogonal system of parametric curves, and that its divergence is equal to the second (or specific) curvature, K, of the surface. Since $\boldsymbol{h}$ is itself tangential to the surface, and by (4) also curl $\boldsymbol{h}$, it follows that $\boldsymbol{h}$ is the gradient of some scalar ‡ function of ϕ. Thus

$$\boldsymbol{h} = \nabla \phi$$

and consequently

$$(5) \qquad K = \operatorname{div} \boldsymbol{h} = \nabla^2 \phi.$$

Again the above analysis is reversible, so that if (4) is satisfied the parametric curves constitute an isometric system. Hence the theorem:

A necessary and sufficient condition that an orthogonal system of curves on a surface may be isometric is that the tangential component of the vector

---

* *Ibid.*, § 8.

† "Some New Theorems in Geometry of a Surface," § 5. *The Mathematical Gazette* (London), Vol. 13 (1926), pp. 1-6.

‡ *Quarterly Journal, loc. cit.*, p. 258.

curvature of the system be the gradient of some scalar point-function, ϕ, on the surface. The second curvature of the surface is then equal to $\nabla^2\phi$.

The value of this function, ϕ, when the condition (1) is satisfied, may be found as follows. The vector $\boldsymbol{h}$ is given by

$$-\boldsymbol{h} = \frac{G_1\boldsymbol{r}_1}{2EG} + \frac{E_2\boldsymbol{r}_2}{2EG} \quad .$$

Now, with the usua notation, E and G may be expressed *

$$E = \lambda U, \qquad G = \lambda V$$

where U is a function of u only, and V a function of v only. Consequently, on differentiation we have

$$E_2/E = G_2/G - V_2/V.$$

Thus the above expression for $-\boldsymbol{h}$ is equivalent to

$$\begin{aligned} -\boldsymbol{h} &= \tfrac{1}{2}(\nabla \log G - \nabla \log V) \\ &= \tfrac{1}{2}\nabla \log \lambda \end{aligned}$$

Thus

(6) $$\phi = -\tfrac{1}{2} \log \lambda$$

and the second curvature of the surface is given by

(7) $$K = -\tfrac{1}{2}\nabla^2 \log \lambda.$$

Cor. For any isometric orthogonal system on a developable surface,

$$\nabla^2 \log \lambda = 0.$$

2. Consider next an orthogonal system of curves cutting an isometric system at a constant angle. We have shown † that the vector curvatures of these two systems are the same. Hence the condition of isometry is satisfied by the new system also; so that:

Any orthogonal system of curves on a surface, cutting an isometric system at a constant angle, is also isometric.

A still more general theorem may be proved as follows. We have shown elsewhere ‡ that the vector curvatures of two orthogonal systems, cutting at a

---

* See the author's *Differential Geometry*, § 39 (Cambridge University Press, 1926).

† *The Mathematical Gazette, loc. cit.*, § 6.

‡ *Ibid.*, p. 6.

variable angle θ, differ by the tangential vector curl $\theta\boldsymbol{n}$ or $\nabla\theta \times \boldsymbol{n}$. If then the condition of isometry is satisfied by one of the systems, it will also be satisfied by the other provided $\nabla\theta \times \boldsymbol{n}$ is the gradient of some scalar function. This will be the case provided *

$$\boldsymbol{n} \cdot \operatorname{curl}\,(\nabla\theta \times \boldsymbol{n}) = 0$$

that is

$$\boldsymbol{n} \cdot \{\boldsymbol{n} \cdot \nabla\nabla\theta - \nabla\theta \cdot \nabla\boldsymbol{n} + (\nabla\theta)\operatorname{div}\boldsymbol{n} - \boldsymbol{n}\,\nabla^2\theta\} = 0.$$

The first three terms vanish identically, showing that the required condition is

$$\nabla^2\theta = 0.$$

Hence the theorem:

An orthogonal system of curves, cutting an isometric orthogonal system at a variable angle θ, will itself be isometric provided $\nabla^2\theta = 0$.

The preceding theorem is a particular case of this.

3. Another form may be found for the condition of isometry of an orthogonal system. If such a system be taken for parametric curves, with $\boldsymbol{a}$ and $\boldsymbol{b}$ as unit tangents, it may be verified that

$$\boldsymbol{b} \cdot \nabla^2\boldsymbol{a} = \frac{1}{(EG)^{1/2}}\left(-JM - \tfrac{1}{2}\,\frac{\partial^2}{\partial u\,\partial v}\,\log\,\frac{E}{G}\right)$$

where L, M, N are the second order magnitudes for the surface, and J is the first curvature, commonly called the mean curvature. Now $M/(EG)^{1/2}$ has the value $\boldsymbol{a} \cdot \operatorname{curl}\boldsymbol{a}$, being equal to the geodesic torsion of the parametric curve $v = \text{const.}$ Hence the above equation shows that the orthogonal system will be isometric provided

$$\boldsymbol{b} \cdot \nabla^2\boldsymbol{a} + J\,\boldsymbol{a} \cdot \operatorname{curl}\boldsymbol{a} = 0.$$

Since $\boldsymbol{b} = \boldsymbol{n} \times \boldsymbol{a}$ we may state the theorem:

A necessary and sufficient condition, that a family of curves with unit tangent $\boldsymbol{t}$ may form with their orthogonal trajectories an isometric system on the surface, is that $\boldsymbol{n} \times \boldsymbol{t} \cdot \nabla^2\boldsymbol{t} + J\,\boldsymbol{t} \cdot \operatorname{curl}\boldsymbol{t}$ vanish identically.

Now the second term of this expression is zero when the curves are lines of curvature on the surface. Therefore:

If $\boldsymbol{a}$, $\boldsymbol{b}$ are unit vectors in the principal directions, a necessary and sufficient condition that the lines of curvature may form an isometric system of curves is that $\boldsymbol{b} \cdot \nabla^2\boldsymbol{a}$ vanish identically.

---

* *Quarterly Journal, loc. cit.*, § 13.

When the lines of curvature form an isometric system, the surface is sometimes said to be *isothermic*. The equation $\boldsymbol{b}\cdot\nabla^2\boldsymbol{a}=0$ may therefore be regarded as the differential equation satisfied by an isothermic surface.

Isometric Families of Surfaces.

4. Consider a triply orthogonal system of surfaces

$$u(x,y,z)=\text{const.},\qquad v(x,y,z)=\text{const.},\qquad w(x,y,z)=\text{const.}$$

Then, in the space occupied by the system, the position vector $\boldsymbol{r}$ of a point is a function of the variables u, v, w. Let differentiation with respect to these three parameters be denoted by the suffixes 1, 2, 3 respectively, and let

$$a=\boldsymbol{r}_1^2,\qquad b=\boldsymbol{r}_2^2,\qquad c=\boldsymbol{r}_3^2.$$

We may take as our starting point the known property that the family of surfaces $w=\text{const.}$ will be isometric provided $\nabla^2 w/(\nabla w)^2$ is a function of w alone, the differential invariants being three-parametric. Now

$$\nabla^2 w=\frac{1}{(abc)^{1/2}}\,\frac{\partial}{\partial w}\left(\frac{ab}{c}\right)^{1/2}$$

and

$$(\nabla w)^2=1/c,$$

so that

$$\frac{\nabla^2 w}{(\nabla w)^2}=\left(\frac{c}{ab}\right)^{1/2}\frac{\partial}{\partial w}\left(\frac{ab}{c}\right)^{1/2}=\frac{\partial}{\partial w}\log\left(\frac{ab}{c}\right)^{1/2}$$

Thus the necessary and sufficient conditions that the family of surfaces $w=$ const. may be isometric are

$$\text{(1)}\qquad \frac{\partial^2}{\partial u\partial w}\log\frac{ab}{c}=0,\qquad \frac{\partial^2}{\partial v\partial w}\log\frac{ab}{c}=0.$$

We shall express these conditions in terms of differential invariants of the unit vector $\boldsymbol{n}$ normal to the surface of the family through the point considered. This unit normal is equal to $\boldsymbol{r}_3/(c)^{1/2}$, so that

$$\operatorname{curl}\boldsymbol{n}=\frac{1}{2c(ab)^{1/2}}\cdot(c_2\boldsymbol{r}_1-c_1\boldsymbol{r}_2)$$

and therefore

$$\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}=\tfrac{1}{2}\left(\frac{c_1\boldsymbol{r}_1}{ac}\right)+\tfrac{1}{2}\left(\frac{c_2\boldsymbol{r}_2}{bc}\right).$$

Also

$$\operatorname{div}\boldsymbol{n}=\frac{1}{(abc)^{1/2}}\,\frac{\partial}{\partial w}\,(ab)^{1/2}$$

and therefore

$$\operatorname{curl}\,(\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n})$$

$$=\frac{1}{2(abc)^{1/2}}\left(\boldsymbol{r}_1\frac{\partial^2}{\partial v\partial w}\log\frac{ab}{c}-\boldsymbol{r}_2\frac{\partial^2}{\partial u\partial w}\log\frac{ab}{c}\right).$$

which vanishes if the family of surfaces is isometric. Conversely, if this vector vanishes identically, the family of surfaces is isometric. Hence the theorem:

A necessary and sufficient condition that a family of surfaces with unit normal $\boldsymbol{n}$ *be isometric is*

$$\text{(2)}\qquad \operatorname{curl}\,(\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n})=0$$

or

$$\operatorname{curl}\,(\boldsymbol{n}\operatorname{div}\boldsymbol{n}-\boldsymbol{n}\cdot\nabla\boldsymbol{n})=0.$$

If this condition is satisfied, the vector $\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}$ is the gradient of some scalar function ϕ. Denoting the vector by $\boldsymbol{H}$, we may write $\boldsymbol{H}=\nabla\phi$. Also we have shown elsewhere * that the second curvature K of a surface of the family is given by

$$2K=\operatorname{div}\,(\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}).$$

Therefore, when the family is isometric, it follows that

$$\text{(3)}\qquad 2K=\operatorname{div}\boldsymbol{H}=\nabla^2\phi$$

so that K is half the Laplacian of ϕ.

The value of the function ϕ, when the conditions (1) are satisfied, may be found as follows. The vector $\boldsymbol{H}$ may be expressed

$$\boldsymbol{H}=\left(\frac{1}{2c}\right)\left(\frac{a_3}{a}+\frac{b_3}{b}\right)\boldsymbol{r}_3+\left(\frac{1}{2c}\right)\left(\frac{c_1}{a}\boldsymbol{r}_1+\frac{c_2}{b}\boldsymbol{r}_2\right)$$

$$=\tfrac{1}{2}\left(\frac{\boldsymbol{r}_1}{a}\frac{\partial}{\partial u}\log c+\frac{\boldsymbol{r}_2}{b}\frac{\partial}{\partial v}\log c+\frac{\boldsymbol{r}_3}{c}\frac{\partial}{\partial w}\log c\right)$$

$$+\tfrac{1}{2}\frac{\boldsymbol{r}_3}{c}\left(\frac{a_3}{a}+\frac{b_3}{b}-\frac{c_3}{c}\right)$$

$$=\tfrac{1}{2}\nabla\log c+\tfrac{1}{2}\frac{\boldsymbol{r}_3}{c}\frac{\partial}{\partial w}\log\frac{ab}{c}.$$

Now $\partial\log(ab/c)/\partial w$ is a function of w alone. Denote it by $f(w)$, and put

$$\int f(w)\,dw=\log\psi(w)$$

* See §2 of a paper by the author "On Families of Curves and Surfaces," recently communicated to the *Messenger of Mathematics.*

so that

$$\psi(w) = e^{\int f(w)\,dw}.$$

Then

$$\boldsymbol{H} = \tfrac{1}{2} \nabla \log c + \tfrac{1}{2} \nabla \log \psi$$

(4) $$= \tfrac{1}{2} \nabla \log (c\psi)$$

and therefore

(5) $$K = \tfrac{1}{4} \nabla^2 \log (c\psi).$$

5. Two consequences of the above theory may be noticed in passing. We have shown elsewhere * that, for a family of *parallel surfaces,* curl $\boldsymbol{n}$ vanishes identically. Hence, if a family of parallel surfaces is also isometric, the equation (2) shows that $\boldsymbol{n} \times \nabla J = 0$. From this it follows that ∇J is parallel to $\boldsymbol{n}$, so that J is constant over any one surface. Hence the theorem:

If a family of parallel surfaces is also isometric, the first curvature is constant over each surface of the family.

Next consider a triply orthogonal system of surfaces, and suppose that the families $v =$ const. and $w =$ const. are both isometric. Then we have the necessary relations

(6) $$\left\{\begin{array}{llll} & \dfrac{\partial^2}{\partial u \partial v} \log \dfrac{ac}{b} = 0, & & \dfrac{\partial^2}{\partial w \partial v} \log \dfrac{ac}{b} = 0 \\ \text{and} & & & \\ & \dfrac{\partial^2}{\partial u \partial w} \log \dfrac{ab}{c} = 0, & & \dfrac{\partial^2}{\partial v \partial w} \log \dfrac{ab}{c} = 0 \end{array}\right.$$

From the second and fourth of these it follows that

(7) $$\frac{\partial^2}{\partial v \partial w} \log a = 0, \qquad \frac{\partial^2}{\partial v \partial w} \log \frac{b}{c} = 0.$$

Now, on a surface $u =$ const., v and w are the current parameters; and the second equation (7) shows that the parametric curves on this surface, which are also lines of curvature by Dupin's theorem, form an isometric system. Hence the theorem:

If two families of a triply orthogonal system of surfaces are isometric, the lines of curvature on any surface of the other family constitute an isometric system of curves.

6. *Congruence of Curves.* We may define an *isometric* congruence of curves as a congruence which cuts orthogonally an isometric system of sur-

---

* *Ibid.*, § 3.

faces. It is therefore first of all a normal congruence, so that the unit tangent $\boldsymbol{t}$ to the congruence satisfies the equation

$$\boldsymbol{t} \cdot \operatorname{curl} \boldsymbol{t} = 0.$$

Then, since $\boldsymbol{t}$ is the unit normal to the isometric family of surfaces, it also satisfies the further condition

$$\operatorname{curl} \left(\boldsymbol{t} \operatorname{div} \boldsymbol{t} - \boldsymbol{t} \cdot \nabla \boldsymbol{t}\right) = 0.$$

These are the equations satisfied by the unit tangent to an isometric congruence.

The vector $(\boldsymbol{t} \operatorname{div} \boldsymbol{t} - \boldsymbol{t} \cdot \nabla \boldsymbol{t})$ plays an important part for any congruence of curves. We have elsewhere* defined the *limit surface* of any congruence, by analogy with that of a rectilinear congruence, and have found its equation in terms of oblique curvilinear coordinates. We here announce (without proof) for the first time the following theorem giving the equation of this surface in terms of differential invariants of the unit tangent:

For a congruence of curves with unit tangent $\boldsymbol{t}$ *the equation of the limit surface may be expressed*

$$\operatorname{div} \left(\boldsymbol{t} \operatorname{div} \boldsymbol{t} - \boldsymbol{t} \cdot \nabla \boldsymbol{t}\right) = 0$$

or

$$\operatorname{div} \left(\boldsymbol{t} \operatorname{div} \boldsymbol{t} + \boldsymbol{t} \times \operatorname{curl} \boldsymbol{t}\right) = 0.$$

Thus, in the case of a normal congruence of curves, the limit surface is the locus of the points at which the second curvature of the family of surfaces orthogonal to the congruence is zero. The *surface of striction* of the congruence is the locus of points at which the first curvature of the family is zero; for this surface is given by $\operatorname{div} \boldsymbol{t} = 0$.

CHRISTCHURCH, N. Z.,
July, 1926.

* "On Congruences of Curves," § 4, recently communicated to the *Tôhoku Mathematical Journal;* also a paper "On Triple Surfaces, etc.", *Proceedings of the Royal Society of Edinburgh,* Vol. 46 (1926), pp. 194-205.

A Characteristic Property of Certain Sets of Trigonometric Functions.

BY M. H. STONE.

We shall discuss in this paper the nature of a set of functions $\{u_i\}$ characterised by certain conditions of orthogonality shared by well-known trigonometric sets, with the aim of showing that the functions u_i are necessarily trigonometric. Before writing down these conditions we state the conventions which we shall follow with regard to the notation for integrals. It is convenient to denote by $\int f$ the integral in the Lebesgue sense of a function $f(x)$ over the interval $(0, 1)$; and only in cases where integrals over other ranges are considered will the limits be indicated explicitly. The functions u_i, $(i = 1, 2, \cdots)$, are then restricted to satisfy the following conditions:

(1) the set $\{u_i\}$ is a closed normal orthogonal set of real functions of the real variable x on the interval $(0, 1)$;

(2) the set $\{v_i\}$, where $v_i = u_i'$, is a set of orthogonal functions on the interval $(0, 1)$;

(3) the functions u_i'' are summable with summable square;

(4) the functions u_i satisfy one of the five sets of boundary conditions

$$
\begin{aligned}
&\text{(a)}\quad u_i(0) = u_i(1) = 0,\\
&\text{(b)}\quad v_i(0) = v_i(1) = 0,\\
&\text{(c)}\quad v_i(0) - v_i(1) = 0, \quad u_i(0) - u_i(1) = 0,\\
&\text{(d)}\quad v_i(0) + v_i(1) = 0, \quad u_i(0) + u_i(1) = 0,\\
&\text{(e)}\quad \alpha_1 v_i(0) - \beta_1 v_i(1) = 0, \quad \alpha_2 u_i(0) - \beta_2 u_i(1) = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha_1^2 - \beta_1^2 \neq 0.
\end{aligned}
$$

Our work is based on the following two lemmas.

LEMMA I. *The set of positive constants ρ_i defined by the equation $\int v_i^2 = \rho_i^2$ has $+\infty$ as its only limit point.*

Suppose that a sequence of values of i can be found such that $\rho_i \to \rho$, $0 \leq \rho < +\infty$, $\int v_i^2 \to \rho^2$. Then, by Schwarz's inequality,

$$| u_i(x) - u_i(y) | = | \int_y^x v_i | \leq (| x - y | \ | \int_y^x v_i^2 |)^{1/2}$$
$$\leq (| x - y | \int v_i^2)^{1/2} \leq M | x - y |^{1/2}$$

for values of i in this sequence. Consequently, the set of functions u_i constitutes a uniformly bounded equicontinuous family. From this family it is possible to select a subsequence converging uniformly to a continuous function $\theta(x)$ on $(0, 1)$. For fixed x we find

$$\int_0^x u_i \to \int_0^x \theta, \qquad \int_0^x u_i \to 0$$

for functions in this subsequence, the last limit being a consequence of Bessel's inequality:

$$\int_0^x 1 \geq \sum_1^\infty (\int_0^x u_i)^2.$$

It follows that $\int_0^x \theta = 0$ whence $\theta \equiv 0$. This leads at once to a contradiction by virtue of the orthogonality of the set $\{u_i\}$. We let i be a value in the sequence $\{i\}$ under consideration; then

$$\int u_i^2 = 1, \qquad \int u_i^2 \to \int \theta^2 = 0 .$$

LEMMA II. *The functions*

$$\phi_i(x; f) = \int_0^x \sin \rho_i(x - y)\, f(y) dy,$$
$$\psi_i\ (x; f) = \int_0^x \cos \rho_i(x - y)\, f(y) dy,$$

where $f(x)$ is summable on $(0, 1)$, converge uniformly to zero with $1/i$.

We give the proof for the function ϕ_i, that for ψ_i being similar. Since

$$| \phi_i(x) - \phi_i(y) | = | \int_y^x \sin \rho_i(x - y)\, f(y) dy | \leq | \int_y^x |f| \, dy | < \epsilon$$

if $| x - y | < \delta$, where δ depends only on f and ϵ, the family of functions $\{\phi_i\}$ is equicontinuous. For each value of x on $(0, 1)$ we have

$$\phi_i = \sin \rho_i x \int_0^\infty \cos \rho_i y\, f(y)\, dy - \cos \rho_i x \int_0^\infty \sin \rho_i y\, f(y)\, dy = o(1)$$

by the fact that when ρ_i becomes infinite the two integrals in the middle term are $o(1)$. Because of the equicontinuity of the family $\{\phi_i\}$ it follows that $\phi_i(x) = o(1)$ uniformly on $(0,1)$.

Lemma I shows that with a suitable renumbering of the set $\{u_i\}$ the inequalities $0 \leq \rho_1 \leq \rho_2 \leq \cdots$ may be supposed valid. In our further reasoning such a renumbering will be assumed.

We can now prove the

THEOREM. *The set $\{u_i\}$ satisfying conditions* (1)–(4) *is one of the closed normal orthogonal sets of trigonometric functions defined by a self-adjoint differential system of the form*

$$u'' + \rho^2 u = 0, \quad \alpha_1 u^{(k_1)}(0) + \beta_1 u^{(k_1)}(1) = 0,$$

$$\alpha_2 u^{(k_2)}(0) + \beta_2 u^{(k_2)}(1) = 0, \quad 1 \geq k_1 \geq k_2 \geq 0,$$

with linearly independent boundary conditions, and conversely.

It should be noticed that the differential system does not define a unique normal orthogonal set because of the fact that in any normal orthogonal set any of the functions may be replaced by its negative. Indeed, in the case corresponding to the boundary conditions (4c) and (4d) the situation is even more complicated.

We have

$$\rho_k^2 \int u_i u_k = \int u_i' u_k' = u_i(1) u_k'(1) - u_i(0) u_k'(0) - \int u_i u_k''$$

whence

$$\text{(A)} \qquad \int (u_k'' + \rho_k^2 u_k) u_i = u_i(1) u_k'(1) - u_i(0) u_k'(0) = a_{ik}.$$

This equation is fundamental in our succeeding work.

When the boundary conditions satisfied by the set $\{u_i\}$ are of types (4a), (4b), (4c), (4d), the constants a_{ik} in (A) all vanish. The closure of the set $\{u_i\}$ then requires that $u_k'' + \rho_k^2 u_k = 0$ almost everywhere on $(0,1)$ for $k = 1, 2, \cdots$. Thus the set $\{u_i\}$ must be one of the normal orthogonal sets determined by the differential equation $u'' + \rho^2 u = 0$ with the boundary conditions (4a), (4b), (4c), or (4d). For special choices of k_1, k_2, α_1, β_1, the system written down in the theorem is equivalent to one or another of these

systems. With (4a) we have $u_i = \pm 2^{1/2} \sin \pi i x$; with (4b), $u_1 = \pm 1$, $u_i = \pm 2^{1/2} \cos \pi(i-1)x$; with (4c),

$$u_1 = \pm 1,$$

$$u_{2l} = \lambda_l \cos 2\pi l x + \mu_l \sin 2\pi l x,$$

$$u_{2l+1} = \mu_l \cos 2\pi l x - \lambda_l \sin 2\pi l x, \quad \lambda_l^2 + \mu_l^2 = 2;$$

with (4d),

$$u_{2l-1} = \lambda_l \cos (2l-1)\pi x + \mu_l \sin (2l-1)\pi x,$$

$$u_{2l} = \mu_l \cos (2l-1)\pi x - \lambda_l \sin (2l-1)\pi x, \quad \lambda_l^2 + \mu_l^2 = 2;$$

When the set $\{u_i\}$ satisfies conditions (4e) the discussion is more difficult. We divide it into three subcases.

Case I. $\beta_1 = 0$. We may assume that $u_k'(1)$ is different from zero for $k = \alpha$; otherwise we fall back on the case in which the set $\{u_i\}$ satisfies the conditions (4b). From (3) and (A) we conclude that

$$\sum_{i=1}^{\infty} u_i^2(1) = \sum_{i=1}^{\infty} a_{i\alpha}^2/u_\alpha'(1)$$

is a convergent series. The theorem of Riesz-Fischer proves the existence of a function f summable with summable square such that $\int f u_i = u_i(1)$; and the closure of the set $\{u_i\}$ enables us to write

$$u_k'' + \rho_k^2 u_k = u_k'(1) f$$

almost everywhere on $(0,1)$. The general solution of this equation is well-known. The function u_k is found to be

$$u_k = a_k \cos \rho_k x + b_k \sin \rho_k x + u_k'(1)\, \phi_k(x; f)/\rho_k,$$

$$u_k' = -\rho_k a_k \sin \rho_k x + \rho_k b_k \cos \rho_k x + u_k'(1)\, \psi_k(x; f),$$

where a_k, b_k are constants depending upon k, and ϕ_k, ψ_k are the functions of Lemma II. The condition $u_k'(0) = 0$ yields $\rho_k b_k = 0$. Since only one of the functions u_i can reduce to a constant, we must have $\rho_k > 0$, $k > 2$, and $b_k = 0$, $k \geqq 2$. Then from Lemma II

$$u_k'(1) = -a_k \rho_k \sin \rho_k + u_k'(1)\, o(1)$$

so that

$$u_k'(1) = -a_k\rho_k \sin \rho_k\,(1 + o(1))$$

and

$$u_k = a_k\,(\cos \rho_k x + o(1)).$$

When this value of u_k is substituted in the equation $\int u_k^2 = 1$, we find

$$1 = a_k^2 \int (\cos \rho_k x + o(1))^2 = a_k^2 \left(\int \cos^2 \rho_k x + o(1)\right)$$
$$= a_k^2 \left[\tfrac{1}{2} + (\sin 2\rho_k)/4\rho_k + o(1)\right] = a_k^2(\tfrac{1}{2} + o(1)).$$

Hence $a_k^2 = 2 + o(1)$, $a_k = \pm 2^{1/2} + o(1)$, and

$$u_k = \pm\, 2^{1/2}\,(\cos \rho_k x + o(1)),$$
$$u_k' = \pm\, 2^{1/2}\,\rho_k\,(-\sin \rho_k x + o(1)).$$

The condition that $\sum u^2_k(1)$ be convergent shows that $\cos \rho_k = o(1)$. We now consider the boundary condition $\alpha_2 u_k(0) + \beta_2 u_k(1) = 0$. On replacing $u_k(0)$, $u_k(1)$ by their asymptotic forms this condition becomes

$$\pm\, 2^{1/2}\,\alpha_2 + \alpha_2\, o(1) + \beta_2\, o(1) = 0$$

so that $\alpha_2 = 0$. Thus the boundary conditions (4e) must take the form $u'_k(0) = 0$, $u_k(1) = 0$. In consequence the constants a_{ik} vanish identically, with the result that

$$u_k'' + \rho_k^2\, u_k = 0$$

almost everywhere. The remainder of the reasoning is analogous to that of the first paragraph in the proof of the theorem. It should be noted that the functions v_k satisfy the self-adjoint differential system

$$v_k'' + \rho_k^2\, v_k = 0,$$
$$v_k'(0) = v_k'(1) = 0,$$

and therefore constitute an orthogonal set.

Case II. $\alpha_1 = 0$. By the transformation $x' = 1 - x$ this case becomes Case I.

Case III. $\alpha_1\beta_1 \neq 0$. We may assume that $u_k'(0)$ is different from zero for $k = \alpha$, since otherwise we should fall back on Case I. Since

$$\alpha_1 u_\alpha'(0) + \beta_1 u_\alpha'(1) = 0$$

we have

$$a_{i\alpha} = [\beta_1 u_i(0) + \alpha_1 u_i(1)]\, u_\alpha'(0)/\beta_1$$

so that

$$\sum_{i=1}^{\infty} [\beta_1 u_i(0) + \alpha_1 u_i(1)]^2/\beta_1^2 = \sum_{i=1}^{\infty} a_{i\alpha}^2/u_\alpha'^2(0) \qquad \text{converges.}$$

Hence there exists a function f summable with summable square such that

$$\int f u_i = [\beta_1 u_i(0) + \alpha_1 u_i(1)]/\beta_1 .$$

As a result

$$u_k'' + \rho_k^2 u_k = u_k'(0)\, f,$$

and

$$u_k = a_k \cos \rho_k + b_k \sin \rho_k x + u_k'(0)\, \phi_k(x; f)/\rho_k,$$

$$u_k' = -\rho_k a_k \sin \rho_k x + b_k \rho_k \cos \rho_k x + u_k'(0)\, \psi_k(x; f).$$

From these equations we find $u_k'(0) = \rho_k b_k$, and

$$u_k = a_k \cos \rho_k x + b_k\, (\sin \rho_k x + o(1)),$$

$$u_k = \rho_k\, [-a_k \sin \rho_k x + b_k\, (\cos \rho_k x + o(1))] .$$

The equation $\int u_k^2 = 1$ shows that

$$1 = a_k^2\, (\tfrac{1}{2} + o(1)) + a_k b_k\, o(1) + b_k^2\, (\tfrac{1}{2} + o(1)).$$

Thus a_k, b_k, which are real quantities in view of the fact that $a_k = u_k(0)$, $b_k = u_k'(0)/\rho_k$, must specify a point on the real trace of the conic C_k:

$$x^2\, (\tfrac{1}{2} + o(1)) + xy\, o(1) + y^2\, (\tfrac{1}{2} + o(1)) = 1.$$

For large values of k this is an ellipse lying in the neighborhood of the circle $x^2 + y^2 = 2$. Consequently $|\, a_k \,|$ and $|\, b_k \,|$ are bounded, and

$$\text{(B)} \qquad a_k^2 + b_k^2 = 2 + o(1).$$

By the convergence of the series $\sum\, (\beta_1 u_k(0) + \alpha_1 u_k(1))^2$ we have

$$\beta_1 u_k(0) + \alpha_1 u_k(1) = o(1).$$

The boundary condition of (4e) involving the first derivatives is

$$\alpha_1 u_k'(0) + \beta_1 u_k'(1) = 0.$$

In these equations we substitute the asymptotic values of the functions u_k and u_k', and, on dividing the second by ρ_k, obtain the relations

$$\beta_1 a_k + \alpha_1 \ (a_k \cos \rho_k + b_k \sin \rho_k) = o(1), \tag{C}$$

$$\alpha_1 b_k + \beta_1 \ (-a_k \sin \rho_k + b_k \cos \rho_k) = o(1). \tag{D}$$

We regard these as equations for a_k and b_k. In view of (B) their determinant must be $o(1)$, since otherwise we would conclude that $a_k = o(1)$, $b_k = o(1)$. Hence we find

$$-2\alpha_1\beta_1 - (\alpha_1^2 + \beta_1^2) \cos \rho_k = o(1),$$

$$\cos \rho_k = -2\alpha_1\beta_1 / (\alpha_1^2 + \beta_1^2) + o(1). \tag{E}$$

From the boundary condition $\alpha_2 u_k(0) + \beta_2 u_k(1) = 0$ we derive the equation

$$\alpha_2 a_k + \beta_2 (a_k \cos \rho_k + b_k \sin \rho_k) = o(1). \tag{F}$$

Taking (C) and (F) as simultaneous equations for a_k and b_k, we notice that the determinant of the system must be $o(1)$ by virtue of (B). This determinant is

$$\sin \rho_k \ (\alpha_1\alpha_2 - \beta_1\beta_2).$$

If $\sin \rho_k = o(1)$, then $\cos \rho_k = 1 + o(1)$ or $\cos \rho_k = -1 + o(1)$. In the first case (E) yields $2\alpha_1\beta_1/(\alpha_1^2 + \beta_1^2) = -1$, $\alpha_1 = -\beta_1$; in the second, $2\alpha_1\beta_1/(\alpha_1^2 + \beta_1^2) = 1$, $\alpha_1 = \beta_1$. These possibilities are excluded by the inequality $\alpha_1^2 - \beta_1^2 \neq 0$. Consequently $\sin \rho_k$ is not $o(1)$; and $\alpha_1\alpha_2 - \beta_1\beta_2 = 0$. We may take $\alpha_2 = \beta_1$, $\beta_2 = \alpha_1$, without loss of generality. Under these conditions the constants a_{ik} in (A) all vanish, so that

$$u_k'' + \rho_k^2 u_k = 0, \quad \alpha_1 u_k'(0) + \beta_1 u_k'(1) = 0,$$
$$\beta_1 u_k(0) + \alpha_1 u_k(1) = 0.$$

The determination of the set $\{u_i\}$ is complete. Since it is clear that the functions v_k satisfy the self-adjoint differential system

$$v_k'' + \rho_k^2 v_k = 0, \quad \beta_1 v_k'(0) + \alpha_1 v_k'(1) = 0,$$
$$\alpha_1 v_k(0) + \beta_1 v_k(1) = 0,$$

they constitute an orthogonal set.

Corollary. *Under conditions* (1)–(4), *the set* $\{v_i/\rho_i\}$ *is a closed normal orthogonal set except in the cases* (4b) *and* (4c).

It is evident that, if the boundary conditions (4) be relinquished, we can still obtain some information concerning the asymptotic character of the functions u_i, v_i and the constants ρ_i when i becomes infinite. In fact, by the consideration of proper subcases, we can use the equation (A) to obtain asymptotic forms under all circumstances. The principal result may be stated roughly by asserting that the set u_i and a properly chosen subset $\bar{u}_i$ of one of the normal orthogonal sets described in the theorem above have the relation $u_i - \bar{u}_i = o(1)$. The fact that we have been unable to apply the properties of the set $\{u_i\}$ in such a manner as to obtain more exact information makes it seem useless to give any account of these rather tedious investigations.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

The Normal Probability Function and General Frequency Functions.

By M. H. Stone.

Many years ago Pearson investigated the problem of representing a given frequency function as the sum of a number of normal probability functions or Gaussian frequency functions. The problem was suggested to him by Weldon, who had seen the possibility of resolving a frequency distribution encountered in certain biological researches into the sum of two normal distributions. Pearson gives a method for obtaining a "best" approximation to a given frequency function by a linear combination of n normal probability functions, but does not consider the convergence of the approximating sums when n becomes infinite. In the specific problem raised by Weldon Pearson shows that his method yields curves fitting the given frequency curves in a highly satisfactory manner.*

The object of the present paper is to discuss the approximation of an arbitrary function $f(x)$ on the interval $(-\infty, +\infty)$ by linear combinations of functions of the form $e^{-\lambda(x-\mu)^2}$, where λ and μ are constants, $\lambda > 0$. We obtain three theorems concerning the representation of an arbitrary function in terms of a *given* set of functions of this type. This set may be so chosen as to include the functions determined by Pearson's process for a given frequency function $f(x)$, when those functions are normal probability functions. We do not obtain any information about the convergence of Pearson's sums, however.

In the following pages all functions considered are real functions of the real variable x defined on the infinite interval $(-\infty, +\infty)$ and measurable in the sense of Lebesgue on that range. All integrals are taken in the sense of Lebesgue; and it should be particularly noted that a function is integrable in the sense of Lebesgue if and only if it is absolutely integrable in the sense of Lebesgue.

Our work is based on the Riesz-Fischer theorem for a set of functions

---

* Karl Pearson, *Philosophical Transactions of the Royal Society*, Vol. 185 A (1894), Part I, pp. 71-111.

normal and orthogonal on the infinite interval $-\infty < x < +\infty$,* and on certain theorems concerning the representation of an arbitrary function $f(x)$ by the integral

$$W_\nu(f;x) = (\nu/\pi)^{1/2} \int_{-\infty}^{+\infty} f(y)\, e^{-\nu(y-x)^2}\, dy$$

when ν becomes infinite.

We begin by establishing four lemmas. The first deals with the integral W_ν defined above, and may be stated as follows:

LEMMA I. *If $f(x)$ is an arbitrary function for which the integral $W_\nu(f;x)$ exists for every pair of values (ν, x), $\nu \geq N > 0$, $-\infty < x < +\infty$, then*

$$\lim_{\nu\to\infty} W_\nu(f;x) = f(x)$$

almost everywhere.

Lebesgue has shown that if $f(x)$ is integrable on $(-\infty, +\infty)$ then

$$\lim_{\nu\to\infty} W_\nu(f;x) = f(x)$$

at every point at which $f(x)$ is the derivative of its indefinite integral; that is, almost everywhere. The convergence is uniform on any closed interval completely interior to any range on which $f(x)$ is continuous..† The hypothesis that $f(x)$ is summable on the infinite interval is unnecessary. We let $f(x)$ be a function for which the integral W_ν exists, as stated in the lemma. For instance a sufficient condition for the existence of this integral is that

$$|\, f(x)\, | = O\,(e^{K|x|^\alpha}),\ \alpha < 2,\ K < 0.$$

We choose any value of x for which $f(x)$ is the derivative of its indefinite integral; for it is clear that on any finite interval $f(y)$ is summable, being the product of the summable function $f(y)e^{-\nu(y-x)^2}$ and the continuous function $e^{\nu(y-x)^2}$. We choose also a fixed positive number σ and a value ν_0 greater than either of the numbers N and $1/4\sigma^4$. Then for $\nu \geq \nu_0$ and for each value of y outside the interval $(x-\sigma, x+\sigma)$ the function $(\nu/\pi)^{1/2}e^{-\nu(y-x)^2}$ is a monotonely decreasing function of ν, as is readily seen by computing its derivative with respect to ν. Thus, by choosing a negative quantity A and

---

* Plancherel, *Rendiconti del Circolo Matematico di Palermo*, Vol. 30 (1910), pp. 290-297.

† H. Lebesgue, *Annales de la Faculté des Sciences de Toulouse*, ser. 3, Vol. 1 (1909), pp. 90-91.

a positive quantity B, both sufficiently large in absolute value, we can write the inequalities

$$\begin{aligned} \left| (\nu/\pi)^{1/2} \int_{-\infty}^{A} f(y) e^{-\nu(y-x)^2} dy \right| &\leq (\nu/\pi)^{1/2} \int_{-\infty}^{A} | f(y) | \, e^{-\nu(y-x)^2} dy \\ &\leq (\nu_0/\pi)^{1/2} \int_{-\infty}^{A} | f(y) | \, e^{-\nu_0(y-x)^2} dy \\ &< \epsilon/4, \end{aligned}$$

and, similarly,

$$\left| (\nu/\pi)^{1/2} \int_{B}^{+\infty} f(y) e^{-\nu(y-x)^2} dy \right| < \epsilon/4$$

where ϵ is a preassigned positive number. Then by Lebesgue's results

$$\left| (\nu/\pi)^{1/2} \int_{A}^{B} f(y) e^{-\nu(y-x)^2} dy - f(x) \right| < \epsilon/2$$

for ν sufficiently large. On adding the three inequalities we see that

$$| W_\nu(f; x) - f(x) | < \epsilon$$

if ν is sufficiently great. The lemma is therefore true.

We let $0 < \lambda_1, \lambda_2, \cdots$, indicate the numbers of a monotonely increasing sequence with limit point $+\infty$, and $\mu_1, \mu_2, \cdots$, the numbers of a countable set everywhere dense in the open continuum $-\infty < \mu < +\infty$. We employ henceforth the notations

$$\phi_{ik} = e^{-\lambda_i(x-\mu_k)^2}, \qquad \psi_{ik} = (x - \mu_k) e^{-\lambda_i(x-\mu_k)^2}.$$

It is then possible to prove

Lemma II. *No finite set of the functions ϕ_{ik} or of the functions ψ_{ik} is linearly dependent.*

We are to prove that the identities

$$\sum c_{ik}\phi_{ik} \equiv 0, \qquad \sum d_{ik}\psi_{ik} \equiv 0, \qquad -\infty < x < +\infty,$$

in which the sums are finite sums, imply that the constant coefficients c and d all vanish.

Let $\sum c_{ik}\phi_{ik} \equiv 0$ be a finite sum in which none of the coefficients vanishes. We single out the least value of λ_i occurring in this sum and denote it by λ_0. Among the functions ϕ containing λ_0 there is one containing a

greatest value of μ_k; we denote this function by ϕ_0, its constant coefficient by c_0, and the value of μ_k which it involves by μ_0. Then we have

$$1 + \sum{}' c_{ik}\phi_{ik}/c_0\phi_0 \equiv 0, \qquad c_0 \neq 0,$$

the sum being extended over all the functions except ϕ_0 appearing in the given identity. Now it is easily seen that the ratio ϕ_{ik}/ϕ_0 has the limit zero when x becomes positively infinite; for, either

$$\phi_{ik}/\phi_0 = e^{-\lambda_i(x-\mu_k)^2 + \lambda_0(x-\mu_0)^2} \to 0, \qquad \lambda_i > \lambda_0,$$

or

$$\phi_{ik}/\phi_0 = e^{2\lambda_0(\mu_k-\mu_0)x + \lambda_0(\mu_0^2-\mu_k^2)} \to 0, \qquad \mu_k < \mu_0.$$

As a result, we can obtain a contradiction by letting x become positively infinite in the second form of the identity. This proves the first part of the lemma.

Because of the fact that $\phi'_{ik} = -2\lambda_i\psi_{ik}$, an identity of the form $\sum d_{ik}\psi_{ik}$ implies an identity $\sum c_{ik}\phi_{ik} \equiv C$, where C is a constant and $c_{ik} = -d_{ik}/2\lambda_i$. On letting x become infinite we see that C must vanish, since each of the functions approaches zero with $1/x$. From the first part of the lemma it follows that the constants c, and therefore the constants d, must vanish.

LEMMA III. *If $f(x)$ is a function for which the integrals $\int_{-\infty}^{+\infty} f\,\phi_{ik}dy$, $i, k = 1, 2, \cdots$, exist, then the integral W_ν exists, $\nu \geq \lambda_1$, $-\infty < x < \infty$, and conversely. If $f(x)$ is a function for which $\int_{-\infty}^{+\infty} f\,\phi_{ik}dy = 0$, $i, k = 1, 2, \cdots$, then $f(x) = 0$ almost everywhere.*

To prove the existence of W_ν we use the fact that if two functions f_1, f_2 satisfy the inequality $|f_1| \leq |f_2|$ and if f_2 is summable on $(-\infty, +\infty)$ then f_1 is summable on the same infinite range. For any given value of x we can choose μ_l and μ_m so that $\mu_l < x < \mu_m$. Then it is easily verified that for $\nu \geq \lambda_1$

$$0 < e^{-\nu(y-x)^2} \leq e^{-\lambda_1(y-x)^2} < Ke^{-\lambda_1(y-\mu_l)^2}, \qquad -\infty < y \leq 0,$$
$$0 < e^{-\nu(y-x)^2} \leq e^{-\lambda_1(y-x)^2} < Ke^{-\lambda_1(y-\mu_m)^2}, \qquad 0 \leq y < +\infty.$$

The principle by which the third inequality in each line is obtained may be stated in graphical terms as follows: the curve representing a function of y is monotonely decreasing and asymptotic to the axis of y; if the curve is translated to the left it lies wholly below its original position. It follows that

$\int_{-\infty}^{+\infty} f(y) e^{-\nu(y-x)^2} dy$ exists. Thus the existence of W_ν is established. The converse is trivial.

Suppose that $\int_{-\infty}^{+\infty} f \phi_{ik} dy = 0$, $i, k = 1, 2, \cdots$. We choose x as any value for which $W_\nu(f; x) \rightarrow f(x)$, since this integral exists and has a limit almost everywhere; we choose μ_l and μ_m so that $\mu_l < x < \mu_m$; and we determine a sequence of values μ_k lying on the range $\mu_l < \mu < \mu_m$ and approaching x as a limit. Then the functions ϕ_{ik} corresponding to these values of μ_k satisfy the inequalities

$$0 < \phi_{ik} < \phi_{il}, \; -\infty < y \leqq \mu_l, \quad 0 < \phi_{ik} < \phi_{im}, \quad \mu_m \leqq y < +\infty.$$

We now choose a and b so that

$$\left| \int_{-\infty}^{a} f \phi_{ik} dy + \int_{b}^{+\infty} f \phi_{ik} dy \right| < \int_{-\infty}^{a} | f | \phi_{il} dy + \int_{b}^{+\infty} | f | \phi_{im} dy < \epsilon/3,$$

$$\left| -\int_{-\infty}^{a} f(y) e^{-\lambda_i (y-x)^2} dy - \int_{b}^{+\infty} f(y) e^{-\lambda_i (y-x)^2} dy \right| < \epsilon/3,$$

where ϵ is an arbitrary preassigned positive number. We notice that $f(x)$ is summable on (a, b), by reasoning like that used in a similar situation in Lemma I. Because $e^{-\lambda_i (y-\mu_k)^2}$ converges uniformly to $e^{-\lambda_i (y-x)^2}$ on (a, b) when μ_k approaches x we can take

$$\left| \int_{a}^{b} f(y) \phi_{ik}(y) dy - \int_{a}^{b} f(y) e^{-\lambda_i (y-x)^2} dy \right| < \epsilon/3$$

by choosing μ_k near enough to x. Adding the three inequalities we obtain

$$\left| \int_{-\infty}^{+\infty} f \phi_{ik} dy - \int_{-\infty}^{+\infty} f(y) e^{-\lambda_i (y-x)^2} dy \right| < \epsilon,$$

whence, by our hypothesis,

$$\left| \int_{-\infty}^{+\infty} f(y) e^{-\lambda_i (y-x)} dy \right| < \epsilon.$$

Since ϵ can be chosen arbitrarily small, i and x being fixed, it follows that for each pair of values (λ_i, x) the integral $W_{\lambda_i}(f; x)$ vanishes. On allowing λ_i to become infinite we find

$$W_{\lambda_i}(f; x) \rightarrow 0, \qquad W_{\lambda_i}(f; x) \rightarrow f(x), \qquad f(x) = 0.$$

Since in choosing x we had merely to avoid the points of a set of measure zero, we conclude that $f(x) = 0$ almost everywhere.

LEMMA IV. *If $f(x)$ is a function for which the integrals $\int_{-\infty}^{+\infty} f\,\psi_{ik}dy$, $i, k = 1, 2, \cdots$, exist, then the integral*

$$I_\nu(f;x) = (2\nu^{3/2}/\pi^{1/2}) \int_{-\infty}^{+\infty} f(y)(y-x)e^{-\nu(y-x)^2}dy$$

exists, and conversely; and $W_\nu(f;x)$ exists. If $f(x)$ is a function such that $\int_{-\infty}^{+\infty} f\,\psi_{ik}dy = 0$, $i, k = 1, 2, \cdots$, then $f(x) = c$, where c is a constant, almost everywhere.

The proof of the existence of the integral I_ν is analogous to that of the corresponding proof in Lemma III. The existence of W_ν can then be deduced almost immediately from the inequality

$$0 < e^{-\nu(y-x)^2} < |\,y-x\,|\,e^{-\nu(y-x)^2}, \qquad |\,y-x\,| > 1.$$

We now suppose $f(x)$ such that $\int_{-\infty}^{+\infty} f\,\psi_{ik}dy = 0$, $i, k = 1, 2, \cdots$. For an arbitrary value of x we can show that $I_\nu(f;x) = 0$, $\nu = \lambda_1, \lambda_2, \cdots$, by reasoning similar to that of the last part of the proof of Lemma III. We next choose a value a for which $W_\nu(f;a) \to f(a)$, and an arbitrary value x for which $W_\nu(f;x) \to f(x)$. If we can show that

$$0 = \int_x^a I_\nu(f;x)dx = W_\nu(f;x) - W_\nu(f;a), \qquad \nu = \lambda_1, \lambda_2, \cdots,$$

we have the results

$$W_{\lambda_i}(f;x) - W_{\lambda_i}(f;a) \to 0, W_{\lambda_i}(f;x) - W_{\lambda_i}(f;a) \to f(x) - f(a) = 0.$$

From the last equation we see that $f(x) = f(a)$ for almost all values of x.

In order to establish the last part of the lemma we wish to show, therefore, that the integral $\int_a^b I_\nu(f;x)dx$ may be evaluated by integrating under the sign of integration in I_ν, for fixed ν and arbitrary finite limits a and b. Now for $a \leq x \leq b$

$$|\,y-x\,|\,e^{-\nu(y-x)^2} \leq |\,y-a\,|\,e^{-\nu(y-a)^2}, \qquad -\infty < y \leq y_1,$$

$$|\,y-x\,|\,e^{-\nu(y-x)^2} \leq |\,y-b\,|\,e^{-\nu(y-b)^2}, \qquad y_2 \leq y < +\infty.$$

Thus, for $a \leq x \leq b$,

$$\left| (2\nu^{3/2}/\pi^{1/2}) \int_{-\infty}^{A} f(y)(y-x) e^{-\nu(x-y)^2} dy + (2\nu^{3/2}/\pi^{1/2}) \int_{B}^{+\infty} f(y)(y-x) e^{-\nu(x-y)^2} dy \right|$$

$$\leq (2\nu^{3/2}/\pi^{1/2}) \int_{-\infty}^{A} | f(y) | \, | y-a | \, e^{-\nu(y-a)^2} dy$$

$$+ (2\nu^{3/2}/\pi^{1/2}) \int_{B}^{+\infty} | f(y) | \, | y-b | \, e^{-\nu(y-b)^2} dy$$

$$\leq \epsilon/2(b-a)$$

for A negative and B positive and both sufficiently great in absolute value. Similarly we have for $x = a$ and $x = b$

$$\left| (\nu/\pi)^{1/2} \int_{-\infty}^{A} f(y) e^{-\nu(y-x)^2} dy + (\nu/\pi)^{1/2} \int_{B}^{+\infty} f(y) e^{-\nu(y-x)^2} dy \right| < \epsilon/4$$

if A and B are properly chosen. It is well known that

$$(2\nu^{3/2}/\pi^{1/2}) \int_{a}^{b} \int_{A}^{B} f(y)(y-x) e^{-\nu(y-x)^2} dy \, dx$$

$$= (\nu/\pi)^{1/2} \int_{A}^{B} f(y) \left(e^{-\nu(y-b)^2} - e^{-\nu(y-a)^2}\right) dy,$$

the integration under the integral sign being permissible. Thus we have

$$\left| \int_{a}^{b} I_\nu(f;x) dx - W_\nu(f;b) - W_\nu(f;a) \right| =$$

$$\Bigg| (2\nu^{3/2}/\pi^{1/2}) \int_{a}^{b} \left(\int_{-\infty}^{A} + \int_{B}^{+\infty} \right) f(y)(y-x) e^{-\nu(y-x)^2} dy \, dx$$

$$- (\nu/\pi)^{1/2} \left(\int_{-\infty}^{A} + \int_{B}^{+\infty} \right) f(y) \left(e^{-\nu(y-b)^2} - e^{-\nu(y-a)^2}\right) dy \Bigg| \leq$$

$$(2\nu^{3/2}/\pi^{1/2}) \int_{a}^{b} \left| \left(\int_{-\infty}^{A} + \int_{B}^{+\infty} \right) f(y)(y-x) e^{-\nu(y-x)^2} dy \right| dx + (\epsilon/2)$$

$$< \epsilon.$$

The desired equality follows at once, since ϵ may be taken arbitrarily small.

We next prove

THEOREM I. *If $f(x)$ is summable on every finite interval, then on any finite interval (a, b) a sequence of linear combinations of functions of the type $e^{-\lambda(x-\mu)^2}$ can be determined converging almost everywhere to $f(x)$, the convergence being uniform on any closed interval completely interior to a range on which $f(x)$ is continuous.*

We choose an interval (A, B) including an arbitrary preassigned interval (a, b) in its interior. We let f_1 be identical with f on (A, B) and identically zero elsewhere. Then $W_\nu(f_1; x)$ converges to $f(x)$ almost everywhere on (a, b), the convergence being uniform on any closed interval completely interior to an interval on which $f(x)$ is continuous.*

We divide the interval (A, B) into n equal intervals $\Delta_1, \cdots, \Delta_n$. In each interval we choose a point $x = \mu_i$. Then for y in Δ_i and x in (a, b)

$$\begin{aligned} | e^{-\nu(y-x)^2} - e^{-\nu(x-\mu_i)^2} | &= 2\nu \,| \mu_i - x + \theta(y - \mu_i) |\, | e^{-\nu(\mu_i - x + \theta(y-\mu_i))^2} |\, | y - \mu_i | \\ &\leq \max | 2Xe^{-X^2} |\, \nu^{1/2} (B - A)/n \\ &< (C\nu^{1/2}/n), \qquad 0 < \theta < 1. \end{aligned}$$

by the law of the mean. Hence we can write

$$\begin{aligned} W_\nu(f_1; x) &= \sum_{i=1}^{n} (\nu/\pi)^{1/2} \int_{\Delta_i} f(y)\,dy \cdot e^{-\nu(x-\mu_i)^2} \\ &\quad + \sum_{i=1}^{n} (\nu/\pi)^{1/2} \int_{\Delta_i} f(y) e^{-\nu(y-x)^2} - e^{-\nu(x-\mu_i)^2})\,dy. \end{aligned}$$

The first term is of the form $\sum_{i=1}^{n} C_i^{(n)}\, e^{-\nu(x-\mu_i)^2})$; the second is in absolute value less than $C\nu \int_A^B | f |\, dy/n\pi^{1/2}$. Hence if $\nu = n^\alpha$, $\alpha < 1$, the last term approaches zero on (a, b). In short

$$\sum_{i=1}^{n} C_i^{(n)}\, e^{-n^\alpha(x-\mu_i)^2} = W_{n^\alpha}(f_1; x) + O(n^{\alpha - 1})$$

uniformly for $a \leq x \leq b$. The theorem results immediately.

THEOREM II. *If $f(x)$ is a function whose square is summable on the infinite interval $(-\infty, +\infty)$, then $f(x)$ can be represented by a sequence of sums of the form $\sum c_{ik}\phi_{ik}$ converging essentially uniformly to $f(x)$ on $(-\infty, +\infty)$.*

By Lemma II the set of functions ϕ_{ik} can be normalized and orthogonalized by Schmidt's process * on the interval $(-\infty, +\infty)$. The functions Φ_i so defined are linear combinations of the functions ϕ_{ik} and conversely. In view of this fact and of Lemma III the functions Φ_i constitute a normal

* Lebesgue, *loc. cit.*

† E. Schmidt, *Mathematische Annalen*, Vol. 63 (1907), pp. 442-444.

orthogonal set closed with respect to the class of functions f for which the integrals $\int_{-\infty}^{+\infty} f\Phi_i dy$ exist, this class being the same as that for which the integrals $\int_{-\infty}^{+\infty} f\phi_{ik} dy$ exist. Any function f whose square is summable on $(-\infty, +\infty)$ can be represented by a sequence of sums selected from the set

$$\sum_{i=1}^{n} c_i\Phi_i, \qquad c_i = \int_{-\infty}^{+\infty} f\Phi_i dy,$$

this sequence converging essentially uniformly to $f(x)$ on $(-\infty, +\infty)$, in accordance with the theorem of Riesz-Fischer in the form given to it by Weyl and Plancherel.† This establishes the theorem, since Φ_i is a linear combination of functions ϕ.

For the proof of the third theorem we employ a continuous function with the following properties

$$\begin{aligned} &p(x) > 0, \\ &x^{\alpha_1}p(x) \to c_1 > 0, \qquad x \to +\infty, \qquad \alpha_1 > \tfrac{1}{2}, \\ &x^{\alpha_2}p(x) \to c_2 > 0, \qquad x \to -\infty, \qquad \alpha_2 > \tfrac{1}{2}. \end{aligned}$$

Then p^2 is summable on $(-\infty, +\infty)$, and its integral over that range may be taken equal to unity. We denote by $g(x)$ any function expressible in the form

$$g(x) = \int_{-\infty}^{x} pf\, dy - A\int_{-\infty}^{x} p^2 dy, \qquad A = \int_{-\infty}^{+\infty} pf\, dy;$$

that is, in the form

$$g(x) = \int_{-\infty}^{x} pf_1 dy, \qquad f_1 = f - Ap, \qquad \int_{-\infty}^{+\infty} pf_1 dy = 0.$$

where f and f_1 are functions whose squares are summable on $(-\infty, +\infty)$. If $g(x)$ is an indefinite integral, such that g and g' satisfy the conditions

$$\begin{aligned} &g(x) \to 0, \qquad |x| \to \infty, \\ &g'(x) = O(x^{\beta_1}), \qquad x \to +\infty, \qquad \alpha_1 + \beta_1 < -1, \\ &g'(x) = O(|x|^{\beta_2}), \qquad x \to -\infty, \qquad \alpha_2 + \beta_2 < -1 \end{aligned}$$

then $f_1 = g'/p$ is a function whose square is summable, and such that

$$g(x) = \int_{-\infty}^{x} pf_1 dy = \int_{-\infty}^{x} g' dy.$$

We can now prove

---

* Plancherel, *loc. cit.*

THEOREM III. *Any function $g(x)$ can be represented by a sequence of sums of the form $\sum c_{ik}\phi_{ik}$ converging uniformly on the infinite range $-\infty < x < +\infty$.*

From Lemma II and the fact that ψ_{ik}/p is summable with summable square on $(-\infty, +\infty)$ we can form from the set of functions ψ_{ik}/p a normal orthogonal set Ψ_i/p. The functions Ψ are expressible as linear combinations of the functions ψ and conversely. Consequently the set of functions f for which the integrals $\int_{-\infty}^{+\infty}(f\,\Psi_i/p)\,dy$ exist is identical with the set for which the integrals $\int_{-\infty}^{+\infty}(f\,\psi_{ik}/p)\,dy$ exist. Since the set of equations $\int_{-\infty}^{+\infty}(f\,\Psi_i/p)\,dy = 0$ implies the set $\int_{-\infty}^{+\infty}(f\,\psi_{ik}/p)\,dy = 0$, it implies also that $f(x)$ reduces almost everywhere to a constant multiple of $p(x)$, by Lemma IV. The set of normal orthogonal functions composed of p and the functions Ψ_i/p is closed with respect to the class of all functions f for which the integrals $\int_{-\infty}^{+\infty} f\,p\,dy$, $\int_{-\infty}^{+\infty}(f\,\Psi_i/p)\,dy$ exist. We now employ the theorem of Riesz-Fischer to show that, if f_1 is a function with summable square for which $\int_{-\infty}^{+\infty} p\,f_1\,dy = 0$,

$$\lim_{\nu\to\infty} \int_{-\infty}^{+\infty}(f_1 - \sum_{i=1}^{n} c_i\Psi_i/p)^2 dy = 0,$$

where $c_i = \int_{-\infty}^{+\infty} f_1\Psi_i/p\,dy$. Thus, by Schwarz's inequality,

$$\left| \int_{-\infty}^{x} p[f_1 - \sum_{i=1}^{n} c_i(\Psi_i/p)]\,dy \right| \leqq \left(\int_{-\infty}^{x} p^2 dy \cdot \int_{-\infty}^{x} [f_1 - \sum_{i=1}^{n} c_i(\Psi_i/p)]^2 dy\right)^{\frac{1}{2}}$$

$$\leqq \left(\int_{-\infty}^{+\infty} [f_1 - \sum_{i=1}^{n} c_i(\Psi_i/p)]^2 dy\right)^{\frac{1}{2}} < \epsilon$$

for n sufficiently large. In other terms, the sequence

$$\sum_{i=1}^{n} c_i \int_{-\infty}^{x} (p\cdot\Psi_i/p)\,dy = \sum_{i=1}^{n} c_i \int_{-\infty}^{x} \Psi_i\,dy = \sum c_{ik}\phi_{ik}$$

converges uniformly to $g(x) = \int_{-\infty} pf_1 dy$ on the range $-\infty < x < \infty$.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

On (3, 3) and Higher Point Correspondences.*

By Temple Rice Hollcroft.

1. *Purpose and general definitions.* The classification of $(2, 2)$ point correspondences between two planes was made in 1917 by F. R. Sharpe and Virgil Snyder.† Since then $(2, 3)$ point correspondences have been classified by the same methods.‡ There are also some earlier papers that discuss certain correspondences but make no attempt at classification. References to these are given in the papers on $(2, 3)$ point correspondences.

Point correspondences are said to be classified when all the algebraic curve systems defining such correspondences are found that are not reducible to each other by birational transformations.

In the present paper it is proposed to classify $(3, 3)$, $(2, 4)$, $(3, 4)$ and $(4, 4)$ algebraic point correspondences and give the principal features of each. The $(3, 3)$ correspondences will be treated in some detail, but only the essentially different features of the others will be discussed.

Point correspondences between two planes, both multiple, may be either involutorial or not involutorial. The latter are called general correspondences, the former, compound involutions. In a general $(3, 3)$ point correspondence, to any point P' of the plane (x') correspond three points P_1, P_2, P_3 of the plane (x). The image of P_1 is P' and two other points P_1' and P_1''; that of P_2 consists of P', P_2', P_2'' and that of P_3: P', P_3', P_3'', all distinct points. Going back to (x), to P_1' correspond P_1, P_4, P_5; to P_1'' correspond P_1, P_6, P_7, etc. The correspondence does not close up, that is, starting with an arbitrary point in either plane, any number of points in either plane can be obtained by repeating the transformation. In the case of a $(3, 3)$ compound involution, however, the correspondence closes up after the second application. To a point P' of (x') correspond three points P_1, P_2, P_3 of (x) to each of which

* Presented to the American Mathematical Society, February 26, 1927.

† F. R. Sharpe and V. Snyder, "Types of (2, 2) Point Correspondences Between Two Planes," *Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 409-414.

‡ T. R. Hollcroft, "A Classification of General (2, 3) Point Correspondences Between Two Planes," *American Journal of Mathematics*, Vol. 41 (1919), pp. 5-24; "On (2, 3) Compound Involutions," *American Journal of Mathematics*, Vol. 43 (1921), pp. 199-212.

correspond the original point P' and two other points P_4', P_5', and the images of either P', P_4' or P_5' are the three points P_1, P_2, P_3 of (x). In this case, one point of either plane determines only three points of each plane.

2. *Properties of the general correspondence.* In the general (3, 3) point correspondence, a type is distinguished and defined by two equations, each containing the coordinates of both planes. Independent types are those whose defining equations can not be reduced to those of another by any birational transformation. There are fifteen independent types of general (3, 3) point correspondences.

The defining equations of a general point correspondence between two planes (x) and (x') may be written as two algebraic equations of the form

$$\sum u_k(x)\, u_k'(x') = 0$$
$$\sum v_k(x)\, v_k'(x') = 0.$$

For any set of values of the x_i' $[x_i]$ these equations represent two curves of (x) $[(x')]$ intersecting in three non-basic points which are the images of the point defined by the original set of values of x_i' $[x_i]$. In addition to the point correspondence, the defining equations establish a transformation between the two planes such that to a line C_1' of (x') $[C_1$ of $(x)]$ corresponds a curve $C_n(x)$ $[C_n'(x')]$ of order n and genus p through the basis points of (x) $[(x')]$.*

In any (3, 3) point correspondence, the general features of both planes are the same, differing only because the defining equations of the two planes differ. In the remainder of this section, the general properties will be given for only one plane, but they will apply equally well to the other.

To a point P' of (x') correspond three points of (x) which describe the image curve $C_n(x)$ as P' describes the line C_n'. To the point of intersection of two lines of (x') correspond in (x) three non-basic intersections of the two curves which are the images of these lines. These three image points always lie at the intersections of the two curves of (x) given by the defining equations for the set of values of x_i' determined by the point P' of (x'). The images are distinct points except for points on certain fixed curves.

If two of the three image points of (x) coincide, the point P' lies on the branch-point curve of (x') which will be denoted by L'. The locus of the corresponding coincidences is the coincidence curve of (x) denoted by K. The locus of the third image point in (x) is the residual curve denoted by G. Both K and G are in (1, 1) correspondence with L'. When all three image

* For an explanation of basis points see Hollcroft, *loc. cit.*, (first paper), page 7.

points coincide, K and G have a common tangent at the coincidence and P' is a cusp on L'. The image of L' is K^2G. Aside from basis points, the only singularities of L' are cusps. K and G each have the same genus as L', but, aside from basis points, their singularities are not further restricted.

The following relation exists among the intersections of the coincidence and residual curves with the respective branch-point curves of the two planes. The contacts of K and L increased by the intersections of G and L equal the contacts of K' and L' increased by the intersections of G' and L'.

If a line C_1 meets K and G in i and j points respectively, its image $C_n'(x)$ has i contacts and j intersections with L'. The image of $C_n'(x)$ is C_1 counted three times and a residual curve that cuts C_1 in $2d+i$ points corresponding to the d non-basic double points of C_n' and the i contacts of L' and C_n'.

3. *Types.* The fifteen independent types of general (3, 3) point correspondences will now be enumerated. Each type is established by a set of defining equations as described above. Special cases may be common to two or more types. In the following table, the types are denoted by Roman numerals and opposite each is given the description of the curves constituting the defining equations belonging to that type. The symbol $C_n; jP_i$ represents a curve of order n which has j basis points each of multiplicity i.

Type	$u_k(x)$	$u_k'(x')$	$v_k(x)$	$v_k'(x')$
I	C_1	C_1	C_3	C_3
II	C_1	C_3	C_3	C_1
III	C_1	$C_n; P_{n-3}$	C_3	$C_1; P_1$
IV	C_1	$C_2; P_1$	C_3	$C_2; P_1$
V	C_1	$C_3; 6P_1$	C_3	$C_3; 6P_1$
VI	C_1	$C_9; 8P_3$	C_3	$C_3; 8P_1$
VII	C_1	$C_2; 2P_1$	C_3	$C_3; P_1, P_2$
VIII	$C_1; P_1$	$C_1; P_1$	$C_n; P_{n-3}$	$C_n'; P_n'_{-3}$
IX	$C_1; P_1$	$C_3; 6P_1$	$C_n; P_{n-3}$	$C_3; 6P_1$
X	$C_1; P_1$	$C_3; 8P_1$	$C_n; P_{n-3}$	$C_9; 8P_3$
XI	$C_2; P_1$	$C_2; P_1$	$C_2; P_1$	$C_2; P_1$
XII	$C_2; P_1$	$C_3; 6P_1$	$C_2; P_1$	$C_3; 6P_1$
XIII	$C_3; 6P_1$	$C_3; 6P_1$	$C_3; 6P_1$	$C_3; 6P_1$
XIV	$C_3; 6P_1$	$C_3; 8P_1$	$C_3; 6P_1$	$C_9; 8P_3$
XV	$C_3; 8P_1$	$C_3; 8P_1$	$C_9; 8P_3$	$C_9; 8P_3$

The methods by which the characteristic curves of each type are obtained are similar to those used in the paper on general (2, 3) point correspondences cited in section 1.

4. *Completeness of the classification.* In order to show that the classification is complete, it must be proved that the defining equations of any general $(3, 3)$ point correspondence are birationally equivalent to the defining equations of some one of these fifteen types. To establish a general $(3, 3)$ point correspondence, it is necessary and sufficient that in both planes the two curves given by the defining equations intersect in three and only three non-basic points, provided that in neither plane both the curves given by the defining equations belong to nets or pencils. The six curve systems employed in the above fifteen types are as follows: line and cubics; line pencil and curves of order n with a point of multiplicity $n-3$ at the vertex of the pencil; conics with one basis point; cubics with six basis points; cubics with a double and a simple basis point through both of which pass conics; cubics with eight basis points and curves of order nine with triple points at each of them.

From these six curve systems can be obtained forty-two sets of defining equations of which twenty-seven are either reducible to the remaining fifteen or else do not define a general $(3, 3)$ point correspondence. The latter is true of all those for which in either plane both the defining curves form nets or pencils since in these cases the correspondence is a compound involution. Also when the fact that one of the components of a defining equation forces the other component to be also a pencil, the correspondence established is usually a special case of some type. For examples of this and of sets reducible to other sets by means of birational transformations, see the paper on general $(2, 3)$ point correspondences cited in section 1.

It now remains to be shown that any curve system that has three non-basic intersections is birationally equivalent to some combination in the above six systems. The proof for this is essentially the same as that for $(1, 2)$ and $(1, 3)$ plane involutions and was first used by Bertini * in reducing $(1, 2)$ involutions to three types.

5. *Compound involutions.* Every $(3, 3)$ compound involution can be expressed as a combination of two $(1, 3)$ involutions and one birational transformation.†

Assume that a $(1, 3)$ involution has been established between two planes (y) and (x). Then to a point P of (y) correspond three points P_1, P_2, P_3 of (x) and to each of these points of (x) corresponds the point P of (y).

---

* E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. 8 (1877), pp. 244-286.

† R. Baldus, "Zur Theorie den gegenzeitig mehrdeutigen algebraischen Ebenentransformationen," *Mathematische Annalen*, Vol. 72 (1912), p. 33.

Also to lines of (x) through P_1, P_2, or P_3 correspond rational curves of (y) forming a net and all passing through P. Assume further that a $(1,3)$ involution has been established between the planes (y') and (x') such that to a point P' of (y') correspond the points P_1', P_2', P_3' of (x').

The planes (y) and (y') contain nets of rational image curves through the points P and P' respectively. Since any two rational curves are birationally equivalent, a $(1,1)$ correspondence exists between (y') and (y) such that P corresponds to P' and P' to P and such that all curves in (y') that are images of rational curves of (x') and all curves in (y') that are images of rational curves of (x') are transformed reciprocally one into the other.

Then to each of the three points P_1', P_2', P_3' of (x') corresponds P' of (y') to which corresponds P of (y) to which correspond P_1, P_2, P_3 of (x). Reciprocally $P_1, P_2, P_3 \sim P \sim P' \sim P_1', P_2', P_3'$. Between the planes (x) and (x') there is thus established a $(3,3)$ compound involution.

The curves of either plane are transformed into curves of the other by three transformations, two rational and one irrational. Since the two planes (y) and (y'), are birationally equivalent, they may be considered the same plane in which the two systems are rationally separable. Then the image in (x') of a curve of (x) is obtained by applying to its image in (y) considered as being also in (y') and as retaining its basis points as fixed points of (y') the transformation from (y') to (x'). A similar series of transformations relates (x') to (x).

If two of the three images of a point P' of (x') coincide, P' is on L', the branchpoint curve of (x'). The locus of the corresponding coincidences in (x) is the coincidence curve K and the locus of the third image point is the residual curve G. K is the Jacobian of the net of curves in (x) that are images of the lines of (y). The image of L' is K counted six times and G counted three times. The complete image of either K or G is L'. The curves L', K and G are not in $(1,1)$ correspondence as in the case of general $(3,3)$ point correspondences. The non-basic intersections of K and G are all contacts to each of which correspond three cusps of L'. The three images of each of these three cusps coincide at the point of contact of K and G. The same general relations hold for L, K' and G'.

The non-basic intersections of L and K, of L and G, of L' and K' and of L' and G' are all equal in number and are all contacts. To each contact of L and K correspond a contact of L' and K' and a contact of L' and G'. To the associated contact of L and G corresponds the same contacts of L' and K' and of L' and G' and the image of each of these is the original pair of contacts of L and K and of L and G.

6. *Types.* There are five independent types of (1, 3) point correspondences.* These types are defined as follows:

Type I.	Lines and cubics of a net.
Type II.	Line pencil, vertex P and C_n with $(n-3)$-fold point P.
Type III.	Two nets of conics with one basis point.
Type IV.	Two nets of cubics with six basis points.
Type V.	Pencil of cubics and net of C_9 with $8P_3$ at basis points of pencil.

Fifteen types of (3, 3) compound involutions are obtained by combining the types of (1, 3) involutions in all possible ways. In the following table, the Roman numerals refer to the types of (1, 3) involutions, and the Arabic numerals to the types of (3, 3) compound involutions. Any type of (3, 3) compound involutions is established by combining the (1, 3) involutions in its row and column. The (1, 3) involutions in the row are those relating the planes (x') and (y') and those in the column relate (x) and (y).

	I	II	III	IV	V
I	1	2	3	4	5
II		6	7	8	9
III			10	11	12
IV				13	14
V					15

The proof that these are all the possible independent types of (3, 3) compound involutions follows immediately from the fact that every (3, 3) compound involution can be established by two (1, 3) involutions and that there are but five independent types of (1, 3) involutions.

The characteristic curves of each type can be readily found by the general method outlined in the preceding section.

7. *Pencil cases.* The theorems proved for (2, 2) and (2, 3) compound involutions hold for all compound involutions and may be stated as follows:

I. *The necessary and sufficient condition that a multiple point correspondence be a compound involution is that the image curves in either plane form a net.*

II. *A sufficient condition that a multiple point correspondence be a com-*

* A. M. Howe, "A Classification of Plane Involutions of Order Three," *American Journal of Mathematics*, Vol. 41 (1919), pp. 25-40.

pound involution is that in either plane both components of the curve system defining the image points form pencils.

In accordance with Theorem II, the fifteen types of general (3, 3) point correspondences become compound involutions when the defining equations each have but two homogeneous parameters. In the following tabulation, the pencil form of each general (3, 3) point correspondence type (Roman numeral) is a special case of the (3, 3) compound involution type (Arabic numeral) given with it.

I, II, III, IV, VII, VIII, XI	6
V, IX, XI	8
VI, X	9
XIII	13
XIV	14
XV	15

8 *Cyclic cases.* In Types II, IV and a special case of Type V of (1, 3) involutions, the three image points form a cyclic projectivity of period three.* Those (3, 3) compound involutions formed from these types retain this property of the image points in one or both planes, depending on whether the compound involution is established by one or two (1, 3) involutions of this kind. In a (1, 3) involution there is but one set of three points determined by two lines of the triple plane and forming a cyclic projectivity, but in a compound involution two lines of one plane determine several triads of points in the other plane, each triad forming a cyclic projectivity of period three. The following table shows the number of triads in each plane for a given type of (3, 3) compound involutions. Types involving Type V of (1, 3) involutions are not the most general (3, 3) compound involutions of those types. In other cases the types are entirely general.

Types	*Number of triads in* (x) *plane*	(x') *plane*
2, 4, 5, 11, 12		16
6	$2n'+1$	$2n+1$
7		$2n+1$
8	36	$2n+1$
9	63	$2n+1$
13	36	36
14	63	36
15	63	63

* A. M. Howe, *loc. cit.*, pp. 39-49.

9. (2, 4) *Point Correspondences.* For the general correspondence, the defining equations are of the same form as those of section 2 except that now the curve system of (x) defines four and that of (x') two non-basic points. The essential difference between any two general point correspondences concerns the branch-point curve and its images, so these will be the only features discussed.

Given a pair of equations defining a general (2, 4) point correspondence between the two planes (x) and (x') such that to a point of (x') correspond four points of (x) and to a point of (x) correspond two points of (x'). The curve L is the locus of points of (x) whose two images in (x') coincide, and the complete image of L is the coincidence locus K' counted twice. The curve L' is the locus of points two of whose images in (x) coincide. The complete image of L' is K^2G, where K is the locus of the two coincident points and G is the locus of the two residual non-coincident points. The complete image of K is L' and the complete image of G is L'^2.

When three images of a point P' coincide, P' is a cusp of L' and the triple coincidence lies at a contact of K and G. When a pair of coincidences corresponds to P', P' is a node of L' and the two coincidences lie at two intersections of K and G such that the line joining them is tangent to G at both points of intersection. All four image points can not in general coincide since the curves of the system defining the four points can not in general have four point contact.

The intersections of L with K and G are all contacts and those of L' and K' are as many contacts as the contacts of L and K and as many intersections as the contacts of L and G.

A (2, 4) compound involution is equivalent to the involution product (2, 1) (1, 1) (1, 4). The discussion of these is similar to that previously given. In the classification of (1, 4) involutions, nine independent types were found* and these may be combined with the three types of (1, 2) involutions in twenty-seven ways, so there are twenty-seven independent types of (2, 4) compound involutions.

10. (3, 4) *Point Correspondences.* When the defining equations determine four non-basic points of (x) and three of (x') a general (3, 4) point

* T. R. Hollcroft, "Plane Involutions of Order Four," *American Journal of Mathematics*, Vol. 44 (1922), pp. 163-171. Professor F. R. Sharpe has called my attention to the fact that in the (x) plane, the net of image curves of Type 9 of (1, 4) involutions is reducible to the net of image curves of Type 8 by quadric inversion. Type 9 should therefore be omitted and there remain nine independent types of (1, 4) plane involutions obtained by the method of this paper.

correspondence is established. The curve L' is now the locus of points two of whose images coincide on K and whose third image lies on G. The curve L is the locus of points which have two coincident images on K' and two distinct images on G'. Both nodes and cusps occur on L but L' has only cusps. Certain types have triads of points in (x') forming cyclic involutions of period three similar to those discussed in section 8.

A $(3,4)$ compound involution is equivalent to the involution product $(3,1)(1,1)(1,4)$. Since there are five types of $(1,3)$ and nine types of $(1,4)$ involutions, there are forty-five independent types of $(3,4)$ compound involutions.

11. $(4,4)$ *Point Correspondences.* When the defining equations determine four non-basic points in each plane, a general $(4,4)$ point correspondence is established. Both L and L' now have properties similar to L of the two preceding sections and both K and K', G and G' now have properties similar to those of K' and G' of the two preceding sections.

A $(4,4)$ compound involution is equivalent to the involution product $(4,1)(1,1)(1,4)$. The nine birationally distinct types of $(1,4)$ involutions, when combined each with itself and with each of the others, give forty-five independent types of $(4,4)$ compound involutions.

12. *Types of general* $(2,4)$, $(3,4)$ *and* $(4,4)$ *point correspondences.* There are, respectively, 30, 65 and 78 independent types of general $(2,4)$, $(3,4)$ and $(4,4)$ algebraic point correspondences. These are exhibited in the following table.

In the uppermost row of the table are given the eleven birationally independent types of algebraic curve systems of the (x) plane that define four points. In the column at the left, the first three curve systems are three birationally independent curve systems of the (x') plane that define two points. (The Bertini curve system C_3, $8P_1$; C_6, $8P_2$, when combined with any curve system always defines an involutorial transformation. The system of two conics with two basis points is not an independent $(1,2)$ involution type when combined with two lines, but, in general, it forms independent types when combined with curve systems defining two or more points.)

The independent types of general point correspondences are defined by equations of the form given in section 2 in which the u_k; v_k are the curve systems given in the row and u_k'; v_k' the curve systems given in the column. Each number denotes a type whose curve systems are given in the row and column in which the number is found. If the number has no subscript, the two curve systems occur in the defining equations in the order given, that is,

the first parts of each curve system form the first defining equation and the second parts of each curve system, the second defining equation. For example, Type 1 of general $(2, 4)$ point correspondences is established by two defining equations, the first formed by $C_1(x)$ and $C_1(x')$ and the second by $C_4(x)$ and $C_2(x')$. If the type number has the subscript r, the equations are formed by reversing the order of either set. For example, Type 2 of $(2, 4)$ point correspondences (denoted by 2_r) is established by equations, the first formed by $C_1(x)$ and $C_2(x')$ and the second by $C_1(x')$ and $C_4(x)$. The combination corresponding to a space in which no number is given is birationally equivalent to some combination for which a number is given. All for which numbers are given are birationally independent of all others for which different numbers are given.

After the first three rows, the next six [last eleven] in the column at the left, contain the six [eleven] birationally independent algebraic curve systems that define three [four] points, and in the six [eleven] rows opposite these are found the type numbers corresponding to combinations of curve systems that define independent types of general $(3, 4)$ $[(4, 4)]$ point correspondences. The last curve system $C_2, 2P_1$; C_4, P_3, P_1 defining four points combines only with curve systems defining more than one point to form independent types of point correspondences.

(x) / (x')	$C_1; C_4$	C_n, P_{n-4}; C_1, P_1;	$C_3, 5P_1$; $C_3, 5P_1$;	$C_2; C_2$	$C_3, 8P_1$; $C_{12}, 8P_4$	$C_3, 8P_1$; $C_4, 8P_1$	C_2, P_1; C_3, P_2	C_2, P_1; $C_3, 2P_1$	$C_2, 2P_1$; $C_4, 2P_2$	$C_3, 7P_1$; $C_4, P_2, 6P_1$	$C_2, 2P_1$; C_4, P_3, P_1
$C_1; C_2$	$1, 2_r$		3	4			$5, 6_r$	$7, 8_r$	$9, 10_r$	$11, 12_r$	$28, 29_r$
C_1, P_1; C_n, P_{n-2}	13_r	14	15		16	17		18_r		$19, 20_r$	
$C_2, 2P_1$; $C_2, 2P_1$	21		22	23			24	25	26	27	30
C_1, C_3	$1, 2_r$	3_r	4	5	6_r	7_r	$8, 9_r$	$10, 11_r$	$12, 13_r$	$14, 15_r$	$60, 61_r$
C_1, P_1; C_n, P_{n-3}	16_r	17	18		19	20		21_r		$22, 23_r$	
C_2, P_1; C_2, P_1	24		25	26			27	28	29	30	62
$C_3, 6P_1$; $C_3, 6P_1$	31	32	33	34	35	36	37	38	39	40	63
$C_3, 8P_1$; $C_9, 8P_3$	41_r	42	43		44	45		46_r		$47, 48_r$	
$C_2, 2P_1$; C_3, P_2, P_1	49		50	51			$52, 53_r$	$54, 55_r$	$56, 57_r$	$58, 59_r$	$64, 65_r$
$C_1; C_4$	$1, 2_r$	3_r	4	5	6_r	7_r	$8, 9_r$	$10, 11_r$	$12, 13_r$	$14, 15_r$	$65, 66_r$
C_1, P_1; C_n, P_{n-4}		16	17		18	19		20_r		$21, 22_r$	
$C_3, 5P_1$; $C_3, 5P_1$			23	24	25	26	27	28	29	30	67
$C_2; C_2$				31			32	33	34	35	68
$C_3, 8P_1$; $C_{12}, 8P_4$					36	37		38_r		$39, 40_r$	
$C_3, 8P_1$; $C_4, 8P_1$						41		42_r		$43, 44_r$	
C_2, P_1; C_3, P_2							$45, 46_r$	$47, 48_r$	$49, 50_r$	$51, 52_r$	$69, 70_r$
$C_2, 2P_1$; $C_3, 2P_1$								$53, 54_r$	$55, 56_r$	$57, 58_r$	$71, 72_r$
$C_2, 2P_1$; $C_4, 2P_2$									$59, 60_r$	$61, 62_r$	$73, 74_r$
$C_3, 7P_1$; $C_4, P_2, 6P_1$										$63, 64_r$	$75, 76_r$
$C_2, 2P_1$; C_4, P_3, P_1											$77, 78_r$

13. *Correspondences of multiplicity higher than four.* The chief distinguishing features of general (m, n) point correspondences lie in the singularities and the relations of the images of the branch-point curve, that is, are concerned with the various ways in which image points may coincide. Two general curves of a defining equation can not have more than three consecutive points in common nor more than two simple contacts. This is true because each pair of curves defining a set of points is determined by two independent parameters, the coordinates of the point of the other plane to which this set of points corresponds. These coincidences of image points occur when either m or n is four, so that for m or n greater than four, no new features of this kind are introduced.

Also, since no new features occur in $(1, n)$ involutions for $n > 4$, a $(4, 4)$ compound involution has all the characteristic features of an (m, n) compound involution for all values of m and n.

It has always been assumed that the curves of the defining equations of both general (m, n) point correspondences and $(1, n)$ involutions are the most general curves possible. In any of these, special curves can be chosen for the defining equations that will allow the coincidence of more image points. For example, in the $(1, 4)$ involution determined by lines and a net of quartics, quartics that possess undulations may be chosen to define the four image points, in which case all four image points may coincide.

The number of independent types of (m, n) point correspondences increases very rapidly as the multiplicity of one plane or each of the two planes increases. There is no further relation between the multiplicity and the number of types.

WELLS COLLEGE.

Contributions to the Theory of Conjugate Nets.

By Ernest P. Lane.

1. *Introduction.* The projective differential geometry of a conjugate net on a curved surface was studied by G. M. Green in two memoirs which appeared in the *American Journal of Mathematics* in 1915 and 1916. In these Green took the conjugate net under discussion as the parametric net.

The recent researches of Fubini and others have shown the power of the methods of the absolute calculus in projective differential geometry. In the present paper we follow Green in taking a conjugate net as parametric, and obtain a new canonical form for Green's equations. This we are able to write with covariant derivatives, using the methods of the absolute calculus. Our canonical form is invariant under all transformations of conjugate nets on the surface.

The differential equations of a surface referred to its projective lines of curvature are obtained, as well as the equations of a surface referred to certain other covariantly defined conjugate nets. The invariants of such a net are really invariants of the surface, and we are enabled thus to define projectively certain classes of surfaces which seem to be of considerable interest.

Several integrals geometrically connected with a conjugate net are set up, and some methods of investigation indicated. The unsolved problem of interpreting these integrals geometrically is formulated.

Wilczynski discovered the two *directrices* associated with each point of a surface (now called by Italian geometers the *lines of Wilczynski*) when he was investigating the congruence of intersection of the osculating linear complexes of the two asymptotic curves through a surface point. In our concluding section we have carried out a similar investigation for the osculating linear complexes of the two curves of a conjugate net through a point of a surface.

2. *A Canonical Form of Green's Differential Equations.* Let the homogeneous coordinates $y^{(1)}, \cdots, y^{(4)}$ of a general point on a non-degenerate nonruled surface be given as analytic functions of two independent variables u, v. If the parametric net is conjugate, the four functions y are solutions of a system of differential equations of the form *

$$\begin{aligned} y_{uu} &= a y_{vv} + b y_u + c y_v + d y, \\ y_{uv} &= \qquad\;\; b' y_u + c' y_v + d' y, \end{aligned} \tag{1}$$

* Green, "Projective Differential Geometry of One-Parameter Families of Space Curves, etc.," *American Journal of Mathematics*, Vol. 37 (1915).

whose coefficients satisfy certain integrability conditions, of which the only one that we shall use is

$$(2) \qquad (b + 2c')_v = (2b' - c/a - a_v/a)_u.$$

Besides the invariant coefficient a, we shall have occasion to use also the following invariants:

$$(3) \qquad \begin{aligned} &H = d' + b'c' - b'_u, && K = d' + b'c' - c'_v, \\ &W^{(u)} = 2c'_v + (c/a)_u, && W^{(v)} = 2b'_u - b_v, \\ &\mathfrak{B}' = (1/8a)(4ab' + 2c - a_v), && \mathfrak{C}' = (1/8a)(4ac' - 2ab + a_u), \\ &R = a\mathfrak{B}'^2 + \mathfrak{C}'^2, && \mathfrak{D} = d' + ab'^2 - c'^2 + b'c + bc' + ab'_v - c'_u. \end{aligned}$$

These invariants are absolutely unchanged by the transformation

$$(4) \qquad y = \lambda \bar{y},$$

which transforms the coefficients of system (1) according to the formulas

$$(5) \qquad \begin{aligned} &\bar{a} = a, \quad \bar{b} = b - 2(\lambda_u/\lambda), \quad \bar{c} = c + 2a(\lambda_v/\lambda), \\ &\qquad \bar{d} = (1/\lambda)(-\lambda_{uu} + a\lambda_{vv} + b\lambda_u + c\lambda_v + d\lambda), \\ &\bar{b}' = b' - \lambda_v/\lambda, \quad \bar{c}' = c' - \lambda_u/\lambda, \\ &\qquad \bar{d}' = (1/\lambda)(-\lambda_{uv} + b'\lambda_u + c'\lambda_v + d'\lambda). \end{aligned}$$

The integrability condition (2) shows that there exists a function q such that

$$(6) \qquad q_u = b + 2c' + \tfrac{3}{2}a_u/a - 2R_u/R, \quad q_v = 2b' - c/a + \tfrac{1}{2}a_v/a - 2R_v/R;$$

and equations (5) show that the effect of the transformation (4) on these derivatives of q is given by

$$\bar{q}_u = q_u - 4\lambda_u/\lambda, \qquad \bar{q}_v = q_v - 4\lambda_v/\lambda.$$

Therefore, if we choose

$$\lambda = e^{\frac{1}{4}q},$$

we shall have $\bar{q}_u = \bar{q}_v = 0$. In this way we obtain a canonical form for system (1), whose coefficients, denoted by capital letters, may be calculated by means of equations (5) and (6). These coefficients are expressed in terms of the invariants of system (1) by the following formulas:

$$(7) \qquad \begin{aligned} &A = a, \quad B = -2\mathfrak{C}' - \tfrac{1}{2}a_u/a + R_u/R, \quad C = a(2\mathfrak{B}' + \tfrac{1}{2}a_v/a - R_v/R), \\ &B' = \mathfrak{B}' + \tfrac{1}{2}R_v/R, \qquad C' = \mathfrak{C}' - \tfrac{1}{2}a_u/a + \tfrac{1}{2}R_u/R, \\ &D = \mathfrak{D} - AB'^2 + C'^2 - AB'_v + C'_u - B'C - BC', \\ &D' = H - B'C' + B'_u = K - B'C' + C'_v. \end{aligned}$$

This canonical form is characterized by the conditions

$$B + 2C' + \tfrac{3}{2}A_u/A - 2R_u/R = 0, \quad 2B' - C/A + \tfrac{1}{2}A_v/A - 2R_v/R = 0. \tag{8}$$

The transformation

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v) \tag{9}$$

leaves the parametric net invariant, but changes the coefficients of system (1) according to the formulas

$$\begin{aligned} \bar{a} &= (\psi_v^2/\phi_u^2)a, \quad \bar{b} = (1/\phi_u)(b - \phi_{uu}/\phi_u) \\ &\qquad \bar{c} = (\psi_v/\phi_u^2)(c + a\psi_{vv}/\psi_v), \quad \bar{d} = (1/\phi_u^2)d, \\ &\bar{b}' = (1/\psi_v)b', \quad \bar{c}' = (1/\phi_u)c', \quad \bar{d}' = (1/\phi_u\psi_v)d', \end{aligned} \tag{10}$$

and changes the relative invariants (3) according to the formulas

$$\bar{\mathfrak{B}}' = (1/\psi_v)\mathfrak{B}', \quad \bar{\mathfrak{C}}' = (1/\phi_u)\mathfrak{C}', \quad \bar{R} = (1/\phi_u^2)R, \quad \bar{\mathfrak{D}} = (1/\phi_u\psi_v)\mathfrak{D}, \tag{11}$$

the invariants H, K, $W^{(u)}$, $W^{(v)}$ being cogredient with $\mathfrak{D}$. Since equations (8) are invariant under this transformation, it follows that our canonical form is undisturbed thereby. In fact *the most general transformation of the group*

$$y = \lambda\bar{y}, \quad \bar{u} = \phi(u), \quad \bar{v} = \psi(v)$$

which leaves our canonical form invariant is obtained by placing $\lambda =$ const.

The quadratic form

$$(R/A)(A du^2 + dv^2) \tag{12}$$

is absolutely invariant under the transformations (4) and (9) and vanishes for the asymptotic net. Calculating the Christoffel symbols of the second kind for this form, we find that the second covariant derivatives of a scalar y with respect thereto are given by the formulas

$$\begin{aligned} y_{11} &= y_{uu} - \tfrac{1}{2}y_u(\partial/\partial u)\log R + (A/2)y_v(\partial/\partial v)\log R, \\ y_{12} &= y_{uv} - \tfrac{1}{2}y_u(\partial/\partial v)\log R - \tfrac{1}{2}y_v(\partial/\partial u)\log(R/A), \\ y_{22} &= y_{vv} + (1/2A)y_u(\partial/\partial u)\log R/A - \tfrac{1}{2}y_v(\partial/\partial v)\log(R/A). \end{aligned} \tag{13}$$

Therefore, *when written with these covariant derivatives, our canonical form becomes*

$$\begin{aligned} y_{11} &= Ay_{22} - 2\mathfrak{C}'y_1 + 2A\mathfrak{B}'y_2 + Dy, \\ y_{12} &= \qquad\quad \mathfrak{B}'y_1 + \mathfrak{C}'y_2 + D'y. \end{aligned} \tag{14}$$

It is not difficult to show that *the canonical form* (14) *is invariant under*

all transformations of curvilinear coordinates which leave a conjugate net parametric. In fact, the most general transformation of this kind is

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

subject to the condition

$$\phi_u\psi_u + A\phi_v\psi_v = 0.$$

Carrying out this transformation, we find that system (14) becomes

$$\begin{aligned} \bar{y}_{11} &= \alpha\bar{y}_{22} + \beta\bar{y}_1 + \gamma\bar{y}_2 + \delta y, \\ \bar{y}_{12} &= \qquad\quad \beta'\bar{y}_1 + \gamma'\bar{y}_2 + \delta' y, \end{aligned}$$

wherein dashes indicate derivatives with respect to $\bar{u}$ and $\bar{v}$, and the coefficients are given by the following formulas:

$$(15)\quad \begin{aligned} &\phi_u{}^2\alpha = \psi_v{}^2 A, \quad \Delta\delta = D(\phi_u{}^2 - A\phi_v{}^2) + 4AD'\phi_u\phi_v, \\ &\Delta\beta = 2[\mathfrak{C}'\phi_u(-\phi_u{}^2 + 3A\phi_v{}^2) + A\mathfrak{B}'\phi_v(3\phi_u{}^2 - A\psi_v{}^2)], \\ &\Delta\phi_u\gamma = 2A[\mathfrak{C}'\phi_v\psi_v(3\phi_u{}^2 - A\phi_v{}^2) + \mathfrak{B}'\phi_u\psi_v(\phi_u{}^2 - 3A\phi_v{}^2)], \\ &\Delta\psi_v\beta' = \mathfrak{C}'\phi_u\phi_v(3\phi_u{}^2 - A\phi_v{}^2) + \mathfrak{B}'\phi_u{}^2(\phi_u{}^2 - 3A\phi_v{}^2), \\ &\Delta\gamma' = \mathfrak{C}'\phi_u(\phi_u{}^2 - 3A\phi_v{}^2) + A\mathfrak{B}'\phi_v(-3\phi_u{}^2 + A\phi_v{}^2), \\ &\Delta\psi_v\delta' = -D\phi_u{}^2\phi_v + D'\phi_u(\phi_u{}^2 - A\phi_v{}^2), \quad \Delta = (\phi_u{}^2 + A\phi_v{}^2)^2. \end{aligned}$$

It is now easy to verify that

$$(16)\qquad \beta + 2\gamma' = 0, \qquad \gamma - 2\alpha\beta' = 0,$$

and *these conditions are characteristic of the canonical form* (14).

3. *The Projective Lines of Curvature and their Reciprocals.* The cubic form

$$(17)\qquad (R/A)(A\mathfrak{C}'du^3 - 3A\mathfrak{B}'du^2dv - 3\mathfrak{C}'dudv^2 + \mathfrak{B}'dv^3)$$

is absolutely invariant, just as is the quadratic form (12), and vanishes * for the curves of Darboux. It is evident that $\mathfrak{B}' = 0$ if, and only if, the curves $u =$ const. are curves of Darboux, and $\mathfrak{C}' = 0$ if, and only if, the curves $v =$ const. are curves of Darboux.

The forms (12) and (17) are the forms ϕ_2 and ϕ_3 of Fubini, as may be verified by calculating the ratio † of the discriminant of (17) to the cube of

* Lane, "Bundles and Pencils of Nets on a Surface," *Transactions of the American Mathematical Society*, Vol. 28 (1926), p. 163.

† Fubini and Čech, *Geometria proiettiva differenziale*, Zanichelli (1926), pp. 84-87.

the discriminant of (12). This ratio is a constant. Therefore *the choice of proportionality factor which leads to the canonical form* (14) *is precisely that which yields Fubini's normal coordinates.* It follows that the second differential parameter $\Delta_2 y$, given by

$$\Delta_2 y = (1/R)(y_{11} + Ay_{22}),$$

represents a point on the projective normal n through the point y, while the first differential parameter $\Delta_1 y$, given by

$$\Delta_1 y = (1/R)(y_1^2 + Ay_2^2),$$

represents the points where the reciprocal n' of the projective normal with respect to the quadric of Lie crosses the asymptotic tangents. Therefore the points y_u and y_v are the points where n' crosses the parametric tangents.

Those curves on a surface in which the developables of the congruence of projective normals intersect the surface are called by Fubini the *projective lines of curvature.* The differential equation of these curves may be determined by the following method. The point ϕ defined by

$$\phi = y_{vv} + (1/A)(C' - 2\mathfrak{C}')y_u + (2\mathfrak{B}' - B' + \tfrac{1}{2}A_v/A)y_v + \lambda y$$

is any point (except P_y) on the projective normal n. As P_y varies along a curve C on the surface, the point ϕ describes a curve, and the point $d\phi$ is on the tangent of the latter. If C corresponds to a developable of the projective normal congruence and if P_ϕ is the corresponding focal point of n, then the point $d\phi$ lies on n, so that $d\phi$ is a linear combination of y and ϕ. So we obtain two equations containing λ and dv/du, and when we eliminate λ therefrom we obtain *the differential equation of the projective lines of curvature*

$$ALdu^2 - Mdudv - Ldv^2 = 0, \tag{18}$$

where

$$L = D' + \mathfrak{C}'_v + \mathfrak{B}'_u + 2C'\mathfrak{B}' + 2B'\mathfrak{C}' - 4\mathfrak{B}'\mathfrak{C}',$$

$$\begin{aligned} M = D + A(4B'\mathfrak{B}' - 4\mathfrak{B}'^2 + 2\mathfrak{B}'_v) + A_v\mathfrak{B}' - 4C'\mathfrak{C}' \\ + 4\mathfrak{C}'^2 - 2\mathfrak{C}'_u + \mathfrak{C}'(A_u/A). \end{aligned} \tag{19}$$

Therefore *the parametric conjugate net is the projective lines of curvature in case* $L = 0$, $M \neq 0$. *And the associate conjugate net of the parametric net is the projective lines of curvature in case* $M = 0$, $L \neq 0$.

We shall now determine the developables of the reciprocal of the projective normal congruence. The point ψ defined by

$$\psi = y_u + \mu y_v$$

is any point on the reciprocal n' of the projective normal (except the point y_v). As P_y moves along a curve, the point ψ also describes a curve, and the point $d\psi$ is on the tangent of the latter. Continuing the argument, which is the same as that used in the case of the projective normal congruence, we find that the differential equation of the curves corresponding to the developables of the reciprocal of the projective normal congruence, which we shall call *the reciprocal projective lines of curvature,* is

$$AD'du^2 - Ddudv - D'dv^2 = 0. \tag{20}$$

Moreover, if μ_1 and μ_2 are the roots of the equation

$$D'\mu^2 + D\mu - AD' = 0, \tag{21}$$

then the corresponding points ψ_1, ψ_2 are the focal points of n'.

The geometrical significance of the invariants D and D' now becomes evident. *The reciprocal projective lines of curvature are parametric in case* $D' = 0$, $D \neq 0$. Then the foci of n' are the points y_u and y_v, the focus y_u corresponding to the curve $u = \text{const.}$ And the *reciprocal projective lines of curvature are the associate net of the parametric net in case* $D = 0$, $D' \neq 0$. If $D = D' = 0$, the reciprocal projective lines of curvature are indeterminate and the lines n' all lie in a fixed plane. The class of surfaces for which this happens seems worthy of a more detailed study than we can make here.

The geometrical significance of the invariants B' and C' is easy to discover. The ray of P_y with respect to the parametric net crosses the parametric tangents at the points ρ, σ given by

$$\rho = y_u - C'y, \qquad \sigma = y_v - B'y.$$

The points ρ and y_u coincide if, and only if, $C' = 0$; similarly the points σ and y_v coincide in case $B' = 0$. *The ray of the parametric net and the line n' reciprocal to the projective normal coincide in case* $B' = C' = 0$.

In case $B' = C' = D' = 0$, $D \neq 0$, the surface is such that the ray of the reciprocal projective lines of curvature coincides with the reciprocal of the projective normal. Then we have $y_{uv} = 0$, $H = K = 0$. The ray-points ρ all lie on one curve, and the points σ on another. The developables of the reciprocal of the projective normal congruence consist of cones each having its vertex on one of these curves and passing through the other. Many special cases arise concerning these curves, but we shall not discuss them here.

If the associate of the parametric net is the reciprocal projective lines of curvature, and if the ray of the parametric net is the reciprocal of the projective normal, then $B' = C' = D = 0$, $D' \neq 0$. Therefore we have $H = K$

$=0$, as we should expect, since the ray curves of the parametric net form a conjugate system. Moreover, we have $\mathfrak{D}=0$, so that the parametric net is harmonic.

All of the preceding considerations may be dualized.

4. *Invariant Integrals.* The projective differential theory of a surface referred to its asymptotics has been enriched by the study of certain integrals geometrically connected with the surface. It is the purpose of this section to show how in a similar way to set up some integrals invariantly connected with a conjugate net and its sustaining surface, and to initiate an investigation thereof.

Let Φ be any one of the invariants

$$(Hv')^{1/2},\ (Kv')^{1/2},\ (W^{(u)}v')^{1/2},\ (W^{(v)}v')^{1/2},\quad (D'v')^{1/2},\ (\mathfrak{D})^{1/2},\ (R)^{1/2},\ \mathfrak{C}',\ \mathfrak{B}'v',$$

where $v'=dv/du$. Then Φdu is an absolutely invariant differential, and the integral $\int\Phi du$ is an invariant integral which has a projective geometric relation to the paramteric conjugate net and its sustaining surface.

In particular let $\phi=(Hv')^{1/2}$. Then Euler's equation for the externals of the integral $\int\phi du$ is

$$v''=(H_u/H)v'-(H_v/H)v'^2. \tag{22}$$

These externals contain the parametric net. They contain the asymptotic net if, and only if,

$$[H/(A)^{1/2}]_u=0,\qquad [H/(A)^{1/2}]_v=0.$$

We propose to find the envelope of the osculating planes at P_y of all the curves (22) which pass through P_y. Using the local tetrahedron with vertices at the points $y,\ y_u,\ y_v,\ z$, where, in the notation of system (1), we have placed

$$z=ay_{vv}+cy_v+dy,$$

we find that the equation of the osculating plane of any curve on the surface through P_y is

$$\begin{aligned}(1/a)(a+v'^2)(v'x_2-x_3)&\\ +(2c'v'-bv'-(c/a)v'^2-2b'v'^2+v'')x_4&=0.\end{aligned} \tag{23}$$

Therefore the coordinates of the osculating plane of a curve (22) are given by

$$\begin{aligned}u_1=0,\quad u_2=(v'/a)(a+v'^2),\quad u_3=-(1/a)(a+v'^2),&\\ u_4=v'P-v'^2Q,&\end{aligned} \tag{24}$$

where

(25) $$P = 2c' - b + H_u/H, \qquad Q = c/a + 2b' + H_v/H.$$

And the equations of the required envelope in plane coordinates are

(26) $$u_1 = 0, \quad u_4(u_2^2 + au_3^2) - a(Pu_2u_3^2 + Qu_2^2u_3) = 0.$$

This is a cone of the third class touching the tangent plane along the asymptotic tangents and having three cusp-planes which intersect in the line joining P_y to the point whose coordinates are $(0, -P, -aQ, 4)$. The directions in which the three cusp-planes meet the tangent plane are given by

$$Pu_2^3 - 3aQu_2^2u_3 - 3aPu_2u_3^2 + a^2Qu_3^3 = 0.$$

These coincide with the directions of Segre, which are the conjugates of the directions of Darboux given by (17), in case $P : Q = \mathfrak{C}' : \mathfrak{B}'$, and coincide with the directions of Darboux themselves in case $P : aQ = -\mathfrak{B}' : \mathfrak{C}'$.

It is easy to determine the developables and focal surfaces of the congruence of cusp-axes of the cones (26) by the methods employed in section 3. And dual considerations lead immediately to a curve of order three in the tangent plane, which is the locus of the ray-points of the extremals (22). This curve has a double point at P_y, and a flex-ray. The congruence of flex-rays might be studied.

Finally, it would be desirable to have a geometric interpretation of each of our invariant integrals. It would be equally desirable to have a geometric interpretation for each of the forms Φdu similar to the interpretation given * by Wilczynski for the metric of Fubini, or the interpretations given by Bompiani and Čech for the linear projective element.

5. *Osculating Linear Complexes.* The osculating linear complex at a point P of a space curve is the limit of the linear complex determined by the tangent at P and four neighboring tangents, as each of the four tangents independently approaches the tangent at P. We shall now obtain the equations in line coordinates of the two linear complexes which osculate at a point P_y of our surface the two curves of the parametric conjugate net through P_y.

The coordinates of a point x near P_y and on the curve C_u, or $v =$ const., through P_y are given by an expansion of the form

(27) $$x = y + y_u\Delta u + \tfrac{1}{2}y_{uu}\Delta u^2 + \cdots,$$

of which we shall need the first six terms. By means of the operations of

* Fubini and Čech, *Geometria proiettiva differenziale*, p. 138.

differentiation and elimination it is possible to express every derivative of y as a linear combination of y_{vv}, y_u, y_v, y by an equation of the form

$$\frac{\partial^{h+k}y}{\partial u^h \partial u^k} = \alpha^{(h,k)}y_{vv} + \beta^{(h,k)}y_u + \gamma^{(h,k)}y_v + \delta^{(h,k)}y.$$

Placing $k=0$, $h \geqq 2$, we may calculate the successive coefficients of (27) by means of the recursion formulas

$$\begin{aligned}
&\alpha^{(h,0)} = \alpha_u{}^{(h-1,0)} + \alpha^{(h-1,0)}\alpha^{(1,2)} + a\beta^{(h-1,0)},\\
(28)\qquad &\beta^{(h,0)} = \beta_u{}^{(h-1,0)} + \alpha^{(h-1,0)}\beta^{(1,2)} + b\beta^{(h-1,0)} + b'\gamma^{(h-1,0)} + \delta^{(h-1,0)},\\
&\gamma^{(h,0)} = \gamma_u{}^{(h-1,0)} + \alpha^{(h-1,0)}\gamma^{(1,2)} + c\beta^{(h-1,0)} + c'\gamma^{(h-1,0)},\\
&\delta^{(h,0)} = \delta_u{}^{(h-1,0)} + \alpha^{(h-1,0)}\delta^{(1,2)} + d\beta^{(h-1,0)} + d'\gamma^{(h-1,0)}.
\end{aligned}$$

Using the points y, y_u, y_v, y_{vv} as vertices of a local tetrahedron of reference, with suitably chosen unit point, we find that the local coordinates of P_x are represented by expansions of the form

$$\begin{aligned}
&x_1 = 1 + \tfrac{1}{2}\, d\, \Delta u^2 + \tfrac{1}{6}\, \delta^{(3,0)}\Delta u^3 + \cdots,\\
(29)\qquad &x_2 = \Delta u + \tfrac{1}{2}\, b\, \Delta u^2 + \tfrac{1}{6}\, \beta^{(3,0)})\Delta u^3 + \cdots,\\
&x_3 = \tfrac{1}{2}\, c\, \Delta u^2 + \tfrac{1}{6}\, \gamma^{(3,0)}\Delta u^3 + \cdots,\\
&x_4 = \tfrac{1}{2}\, a\, \Delta u^2 + \tfrac{1}{6}\, \alpha^{(3,0)}\Delta u^2 + \cdots.
\end{aligned}$$

The corresponding expansions for the coordinates of a point z, other than P_x, on the tangent of C_u at P_x, are obtained by differentiating each of the series (29) with respect to Δu. The Plückerian coordinates ω_{ik} of the tangent of C_u at P_x are defined by the usual formulas,

$$\omega_{ik} = x_i z_k - x_k z_i,$$

and are represented by the following expansions:

$$\begin{aligned}
&\omega_{12} = 1 + \cdots,\\
&\omega_{13} = c\,\Delta u + \tfrac{1}{2}\gamma^{(3,0)}\Delta u^2 + \tfrac{1}{6}\gamma^{(4,0)}\Delta u^3\\
&\qquad\qquad + \tfrac{1}{24}(\gamma^{(5,0)} + 2d\gamma^{(3,0)} - 2c\delta^{(3,0)})\Delta u^4 + \cdots,\\
&\omega_{14} = a\,\Delta u + \tfrac{1}{2}\alpha^{(3,0)}\Delta u^2 + \tfrac{1}{6}\alpha^{(4,0)}\Delta u^3\\
&\qquad\qquad + \tfrac{1}{24}(\alpha^{(5,0)} + 2d\alpha^{(3,0)} - 2a\delta^{(3,0)})\Delta u^4 + \cdots,\\
(30)\qquad &\omega_{23} = \tfrac{1}{2}c\,\Delta u^2 + \tfrac{1}{3}\gamma^{(3,0)}\Delta u^3 + \tfrac{1}{24}(3\gamma^{(4,0)} + 2b\gamma^{(3,0)} - 2c\beta^{(3,0)})\Delta u^4 + \cdots,\\
&\omega_{42} = -\tfrac{1}{2}a\,\Delta u^2 - \tfrac{1}{3}\alpha^{(3,0)}\Delta u^3\\
&\qquad\qquad - \tfrac{1}{24}(3\alpha^{(4,0)} + 2b\alpha^{(3,0)} - 2a\beta^{(3,0)})\Delta u^4 + \cdots,\\
&\omega_{34} = \tfrac{1}{12}(c\,\alpha^{(3,0)} - a\gamma^{(3,0)})\Delta u^4 + \cdots,
\end{aligned}$$

in which we have written as many terms as we shall need. Writing the equation of a linear complex in the form $\sum a_{ik}\omega_{ik}=0$, we determine the coefficients so that this equation may be satisfied by the series (30) identically in Δu as far as the terms in Δu^4. In this way we find

$$
\begin{aligned}
&a_{12}=0,\quad a_{13}=4aA^2,\quad a_{14}=-4cA^2,\\
(31)\qquad &a_{23}=2A(2A\alpha^{(3,0)}-aB'),\quad a_{42}=2A(2A\gamma^{(3,0)}-cB'),\\
&a_{34}=2A[C+2A(d+\beta^{(3,0)})+3(\gamma^{(4,0)}\alpha^{(3,0)}\\
&\qquad\qquad -\alpha^{(4,0)}\gamma^{(3,0)})-bB']-3B'^2,
\end{aligned}
$$

where

$$
\begin{aligned}
&A=a^2(K+W^{(u)}),\\
(32)\qquad &B'=A_u+A(b+2c'),\\
&C=B_u+c'B+(c_u+a\gamma^{(1,2)})\alpha^{(4,0)}-a_u\gamma^{(4,0)}.
\end{aligned}
$$

The coefficients (31) are not all invariants because the vertices of the tetrahedron of reference are not all covariant. If we choose a new tetrahedron of reference with vertices at the points y,ρ,σ,τ for which

$$
\begin{aligned}
&\rho=y_u-c'y,\qquad \sigma=y_v-b'y,\\
&\tau=ay_{vv}+cy_v-(\tfrac{1}{2}\mathfrak{D}+c'^2+c'_u-bc'-d)y,
\end{aligned}
$$

then the points ρ,σ are the ray points of P_y with respect to the parametric net, and the point τ is the harmonic conjugate of P_y with respect to the foci of the axis of P_y. Referred to this tetrahedron, *the equation of the osculating linear complex of the curve C_u is*

$$
\begin{aligned}
(33)\qquad &4aA^2\omega_{13}-2aAI\omega_{23}+4A^3\omega_{42}\\
&\quad +a[2AI_u+16AI(\mathfrak{C}'-\tfrac{3}{8}a_u/a)-3I^2+2A^2\mathfrak{D}]\omega_{34}=0,
\end{aligned}
$$

where

$$
(34)\qquad I=A_u+4A(\mathfrak{C}'-\tfrac{5}{8}a_u/a).
$$

Similarly, it can be shown that *the equation of the osculating linear complex of the curve C_v, referred to the same tetrahedron $y\rho\sigma\tau$, is*

$$
\begin{aligned}
(35)\qquad &4B^2\omega_{12}+3BJ\omega_{23}-4aB^3\omega_{34}\\
&\quad -[2aBJ_v+16aBJ(\mathfrak{B}'+\tfrac{1}{4}a_v/a)-3aJ^2-2B^2\mathfrak{D}]\omega_{42}=0,
\end{aligned}
$$

where

$$
\begin{aligned}
&B=(1/a)(H+W^{(v)}),\\
(36)\qquad &J=B_v+4B(\mathfrak{B}'+\tfrac{3}{8}a_v/a).
\end{aligned}
$$

We shall refer to the complexes (33) and (35) as the first and second complexes respectively.

Some properties of these complexes may now be deduced. If the curve C_u is a plane curve, then $A=0$ and the first complex is indeterminate. The invariant of the first complex $16aA^5$. Therefore *if C_u is not a plane curve, the first complex is not special.* Since the coefficients a_{12} and a_{14} of the first complex are zero, it follows that *this complex contains all the lines of the pencil with center at P_y and lying in the osculating plane of C_u.* In fact, P_y corresponds to the osculating plane of C_u in the null system of the first complex. Moreover, the point ρ corresponds to the plane

$$aIx_3 + 2A^2x_4 = 0.$$

This plane coincides with the tangent plane of the surface if, and only if, $I=0$. In case $I=0$, to P_σ corresponds the plane

$$2x_1 - \mathfrak{D}x_4 = 0,$$

and to P_τ corresponds the plane

$$Ax_2 - a\mathfrak{D}x_3 = 0.$$

If $I=\mathfrak{D}=0$, to P_σ corresponds the plane $\rho\sigma\tau$, and to P_τ corresponds the osculating plane of C_v. Finally, the ray $\rho\sigma$ belongs to the first complex in case $I=0$. Similar statements are true regarding the second complex.

In order to obtain the equations of the directrices of the linear congruence of intersection of the first and second complexes, it is sufficient to write down the conditions that the plane corresponding to a variable point in the null system of a special complex of the pencil determined by the two complexes may be indeterminate. Denoting the coefficients of the first and second complexes by a_{ik} and b_{ik} respectively, we obtain for the equations of the directrices

(37) $$b_{12}x_2 + \lambda_i a_{13}x_3 = 0,$$
$$b_{12}x_1 + (\lambda_i a_{23} + b_{23})x_3 + (a_{42}\lambda_i + b_{42})x_4 = 0,$$

where λ_i $(i=1, 2)$ are the roots of the quadratic

(38) $$a_{13}a_{42}\lambda^2 + (a_{13}b_{42} + a_{34}b_{12})\lambda + b_{12}b_{34} = 0$$

which determines the special complexes of the pencil.

The directrices do not lie in the osculating plane of C_u, or of C_v, but do intersect the axis $y\tau$ in the points

$$(a_{42}\lambda_i + b_{42},\ 0,\ 0,\ -b_{12}), \qquad (i=1,2).$$

If $I=J=0$, the directrices meet the ray $\rho\sigma$ at the points

$$(0,\ \lambda_i a A^2,\ -B^2,\ 0),$$

where λ_i are the roots of the equation

$$2A^5\lambda^2 + A^2B^2\mathfrak{D}\lambda - 2B^5 = 0.$$

In this case the simultaneous invariant of the two complexes vanishes if, and only if, $\mathfrak{D}=0$, so that the two complexes are in involution if, and only if, the parametric net is harmonic.

UNIVERSITY OF ROME,
ROME, ITALY.

A Certain General Type of Neumann Expansions and Expansions in Confluent Hypergeometric Functions.

By R. F. Graesser.

Introduction.

1. *Statement of the Problem.* The purpose of this paper is to obtain two expansion theories analogous to the one developed by Neumann and Gegenbauer for Bessel functions.* In the first division we deal with a certain general sequence of analytic functions of a complex variable z. Each element of this sequence is defined by a power series and all such series are assumed to converge in a single circle of finite radius. With one additional hypothesis, which will be found in § 2, it is possible to obtain an expansion of an analytic function $f(z)$ which will converge uniformly with respect to z in some finite region (Theorem II). An expansion of $f(z)$ is developed in terms of a sequence of polynomials in $1/z$ which are associated with the original sequence of analytic functions. An analogue to Laurent's expansion is obtained. The relation of these expansions to the ordinary Taylor and Laurent expansions of $f(z)$ is shown. Some properties of the elements of the sequence and of their associated polynomials in $1/z$ are found. Generalizations are made to functions of n complex variables.

In the second division a certain confluent hypergeometric equation is subjected to a transformation. Expansions in terms of a solution of the transformed equation are first discussed as a special case of the theory developed in division I. The results are then extended by further investigation.

The expansion theory, which is developed in detail in the present paper, revolves about our Theorems II and III. The latter are related to certain theorems of great generality due to G. D. Birkhoff.† The methods of the

* See Watson, *Theory of Bessel Functions* (London, 1922), Chapters IX and XVI, where an excellent summary of these results is given as well as copious references.

† "Sur une généralisation de la Séries de Taylor," *Comptes Rendu des Séances de l'Académie des Sciences*, Vol. 164 (1917), pp. 942-945.

present paper are different, however, from those used by Birkhoff. Comparison of these theorems should also be made with certain results obtained in a paper by J. L. Walsh.*

I. A Certain General Type of Neumann Expansions.

2. *The Sequence of Functions and the Hypotheses.* We take the general sequence of analytic functions obtained by allowing m to range over zero and the positive integers in the following definition of $F_m(z)$,

$$F_m(z) = \sum_{s=0}^{\infty} c_{ms} z^{m+s}, \quad c_{m0} = 1. \tag{1}$$

The power series defining each function of the above sequence is assumed to be convergent within some circle of radius R described about the origin as a center where R is independent of m. We have a well-known theorem which says:

"If $F(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ is a power series convergent in a circle described about the origin with radius R and if $N < R$ then the greatest value which $|F(z)|$ takes on on the circle of radius N described about the origin is at least as great as $|c_\nu| N^\nu$."

Designate the greatest value which $|F_m(z)|$ takes on on the circle of radius $N < R$ described about the origin by $M_N(m)$; then

$$M_N(m) \geq |c_{ms}| N^{m+s}. \tag{2}$$

We will now make the further hypothesis that a constant M_N exists such that for all values of m we have

$$M_N \geq M_N(m) N^{-m}.$$

3. *Expansion of z^n.* Proceeding formally to compute the coefficients γ_{nm}, $m = 0, 1, 2, \cdots$, of the expansion

$$z^n = \sum_{m=0}^{\infty} \gamma_{nm} F_m(z)$$

we insert in the right member the expansion of $F_m(z)$ and equate coefficients of like powers of z. We then find that the γ's satisfy the recurrence relation

* *Transactions of the American Mathematical Society*, Vol. 26 (1924), pp. 155-170.

(3) $\gamma_{n0}c_{0\nu} + \gamma_{n1}c_{1,\nu-1} + \gamma_{n2}c_{2,\nu-2} + \cdots + \gamma_{n,\nu-1}c_{\nu-1,1} + \gamma_{n\nu} = 1, \quad \nu = n;$
$= 0, \quad \nu \neq n;$

from which we have $\gamma_{n\nu} = 0$, $\nu = 0, 1, 2, \cdots, (n-1)$; $\gamma_{nn} = 1$ and

(4) $$\gamma_{n,n+\nu} = (-)^{\nu+1} \begin{vmatrix} c_{n,1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{n,2} & c_{n+1,1} & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n,\nu-2} & c_{n+1,\nu-3} & c_{n+2,\nu-4} & \cdots & \cdots & 1 & 0 \\ c_{n,\nu-1} & c_{n+1,\nu-2} & c_{n+2,\nu-3} & \cdots & \cdots & c_{n+\nu-2,1} & 1 \\ c_{n,\nu} & c_{n+1,\nu-1} & c_{n+2,\nu-2} & c_{n+3,\nu-3} & \cdots & c_{n+\nu-2,2} & c_{n+\nu-1,1} \end{vmatrix}, \quad \nu > 0.$$

We may then start the summation with n obtaining

(5) $$z^n = \sum_{\nu=0}^{\infty} \gamma_{n,n+\nu} F_{n+\nu}(z).$$

We shall show that the series in the right member is absolutely and uniformly convergent within and upon the circle $|z| = R_1$, where $R_1 < N/(M_N + 1)$, and that it is a valid expansion of z^n in the closed region bounded by this circle. To do this we need dominating expressions for $|F_{n+\nu}(z)|$ and $|\gamma_{n,n+\nu}|$.

Replacing $M_N(m)$ in (2) by its dominant we get

(6) $$|c_{ms}| \leq M_N N^{-s}.$$

Writing $n + \nu$ for m we have from the expansion for $F_m(z)$ the relation

(7) $$|F_{n+\nu}(z)| \leq |z|^{n+\nu} M_N \sum_{s=0}^{\infty} (|z|/N)^s = M_N N |z|^{n+\nu}/(N - |z|)$$

provided $|z| < N$.

When we omit the vanishing terms from (3) that recurrence relation becomes

$$\gamma_{nn}c_{n,\nu-n} + \gamma_{n,n+1}c_{n+1,\nu-n-1} + \cdots + \gamma_{n,\nu-1}c_{\nu-1,1} + \gamma_{n\nu} = 1, \quad \nu = n;$$
$$= 0, \quad \nu > n.$$

By giving ν the values $n, n+1, \cdots$ and employing the inequality (6) we have successively the following relations:

$$
\begin{aligned}
&|\gamma_{n,n}| = 1,\\
&|\gamma_{n,n+1}| \leq N^{-1}M_N,\\
&|\gamma_{n,n+2}| \leq N^{-2}M_N(M_N+1),\\
&|\gamma_{n,n+3}| \leq N^{-3}M_N(M_N+1)^2,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&|\gamma_{n,n+\nu}| \leq N^{-\nu}M_N(M_N+1)^{\nu-1}.
\end{aligned}
$$

Hence

$$
|\gamma_{n,n+\nu}| \leq N^{-\nu}(M_N+1)^{\nu}. \tag{8}
$$

Making use of these two dominants we have

$$
|\gamma_{n,n+\nu}|\,|F_{n+\nu}(z)| \leq \frac{M_N N\,|z|^n}{N-|z|}\left[\frac{(M_N+1)\,|z|}{N}\right]^{\nu}.
$$

The right member is the νth term of a convergent series if $|z| < N/(M_N+1)$ so that the series in (5) is absolutely convergent. If $|z| \leq R_1 < N/(M_N+1)$ we may choose R_2 such that $R_1 < R_2 < N/(M_N+1)$; then we have the relation

$$
|\gamma_{n,n+\nu}|\,|F_{n+\nu}(z)| \leq \frac{M_N N R_2{}^n}{N-R_2}\left[\frac{(M_N+1)R_2}{N}\right]^{\nu}.
$$

The right member is the νth term of a convergent series of positive constants. The absolute and uniform convergence of (5) follows from the test of Weierstrass provided that $|z| \leq R_1 < N/(M_N+1)$.

By another theorem due to Weierstrass (cf. Osgood, *Lehrbuch der Funktionentheorie,* Vol. I, 4th Ed., Leipzig, 1923, p. 343) we may substitute for $F_{n+\nu}(z)$ from (1) in (5) and write the result in ascending powers of z. When this is done it is easily seen by the aid of the recurrence relations (3) on the γ's that the sum of the series in (5) is z^n.

4. *Expansion of* $1/(t-z)$. We have the expansion

$$
1/(t-z) = \sum_{n=0}^{\infty} z^n/t^{n+1}
$$

if $|z| < |t|$. We substitute for z^n its expansion from (5). Then we have

$$
1/(t-z) = \sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} \gamma_{n,n+\nu}F_{n+\nu}(z)/t^{n+1}. \tag{9}
$$

Proceeding formally we arrange the result according to ascending values of the subscripts of the F's and obtain the relation

$$(10) \qquad 1/(t-z) = \sum_{n=0}^{\infty} V_n(t)F_n(z), \quad \text{where } V_n(t) = \sum_{r=0}^{n} \gamma_{r,n}/t^{r+1}.$$

To justify this rearrangement it is sufficient to prove the absolute convergence of the series in (9). (Cf. Hobson, *Theory of Functions of a Real Variable* 1st Ed., Cambridge, 1907, p. 465). Making use of (7) and (8) we have

$$\frac{|\gamma_{n,n+\nu}|\,|F_{n+\nu}(z)|}{|t|^{n+1}} \leq \frac{M_N N\,|z|^n}{(N-|z|)\,|t|^{n+1}}\left[\frac{(M_N+1)\,|z|}{N}\right]^{\nu}$$

for each value of ν and

$$\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{|\gamma_{n,n+\nu}|\,|F_{n+\nu}(z)|}{|t|^{n+1}} \leq \sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{M_N N\,|z|^n}{(N-|z|)\,|t|^{n+1}}\left[\frac{(M_N+1)\,|z|}{N}\right]^{\nu}.$$

The double series in the right member is convergent if $|z| < N/(M_N+1)$ and $|z| < |t|$. Consequently the double series in (9) is absolutely convergent in the open regions defined by the inequalities

$$|z| < N/(M_N+1), \quad |z| < |t|;$$

and (10) is valid in the same regions.

Suppose $|z| \leq R_1 < N/(M_N+1)$ and $|t| \geq R_4$ where $R_1 < R_4$. We can choose R_2 and R_3 such that $R_2 < N/(M_N+1)$ and $R_1 < R_2 < R_3 < R_4$. The series in (9), and therefore the double series obtained from the first equation in (10) by substitution for $V_n(t)$ its value in (10), is dominated by the convergent series of positive constants

$$\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{M_N N R_2^{\,n}}{(N-R_2)R_3^{\,n+1}}\left[\frac{(M_N+1)R_2^{\,n}}{N}\right]^{\nu}.$$

Hence by the theorem of Weierstrass the double series is absolutely and uniformly convergent with respect to both z and t in the regions indicated. We may state the following theorem:

THEOREM I. *Let z and t be two complex variables such that*

$|z| \leq R_1 < N/(M_N+1)$ and $|t| \geq R_4$ where $R_1 < R_4$; then the expansion

$$1/(t-z) = \sum_{n=0}^{\infty} V_n(t)F_n(z), \quad \textit{where} \quad V_n(t) = \sum_{r=0}^{n} \gamma_{rn}/t^{r+1},$$

is valid. If t is a variable point upon or outside the circle $|z| = R_4$ and z is within or upon the circle $|z| = R_1$ then the expansion converges uniformly with respect to t and z.

In the remainder of the paper we shall use the symbols R_1, R_2, R_3, R_4 in a generic sense, the numbers denoted by them varying from section to section. In all cases we shall suppose that $R_1 < R_2 < R_3 < R_4$. Moreover we shall employ the symbol C_i to denote the circle $|z| = R_i$ for $i = 1, 2, 3, 4$.

5. *Expansion of $f(z)$ in Terms of $F_n(z)$.* By means of theorem I and Cauchy's integral formula we are able to expand any function $f(z)$, analytic and single-valued within and upon a circle C_4, the equation of C_4 being $|z| = R_4$; thus,

$$\begin{aligned} f(z) &= (1/2\pi i)\int_{C_4} \frac{f(t)}{t-z}\, dt \\ &= (1/2\pi i)\int_{C_4} \sum_{n=0}^{\infty} V_n(t)F_n(z)f(t)\,dt. \end{aligned}$$

The expansion in the second member is uniformly convergent with respect to t and z under the conditions of theorem I and may therefore be integrated term by term. Hence

$$f(z) = \sum_{n=0}^{\infty} a_nF_n(z), \text{ where } a_n = (1/2\pi i)\int_{C_4} V_n(t)f(t)\,dt.$$

This expansion is valid and converges uniformly with respect to z if $|z| \leq R_1 < R_4$ and $R_1 < N/(M_N + 1)$. Summarizing we have the following theorem:

THEOREM II. *If $f(z)$ be a single-valued, analytic function of z within and upon C_4, the equation of C_4 being $|z| = R_4$, then the expansion,*

$$f(z) = \sum_{n=0}^{\infty} a_nF_n(z), \text{ where } a_n = (1/2\pi i)\int_{C_4} V_n(t)f(t)\,dt,$$

is valid and converges uniformly with respect to z if z is within or upon the circle $|z| = R_1$ where $R_1 < N/(M_N + 1)$ and $R_1 < R_4$.

6. *Expansion of $f(z)$ in Terms of $V_n(z)$.* We shall prove the following theorem:

THEOREM III. *If $f(z)$ be a single-valued, analytic function of z upon*

and outside of a circle C_1, the equation of C_1 being $|z| = R_1$, then the expansion

$$f(z) = \sum_{n=0}^{\infty} \alpha_n V_n(z), \text{ where } \alpha_n = (1/2\pi i) \int_{C_1} F_n(t) f(t)\, dt,$$

is valid and converges uniformly with respect to z if z is upon or outside the circle $|z| = R_4$ where $R_1 < N/(M_N + 1)$ and $R_1 < R_4$.

From theorem I we have

$$1/(t - z) = -\sum_{n=0}^{\infty} F_n(t) V_n(z);$$

this series converges uniformly with respect to z and t if

$$|t| \leq R_1 < N/(M_N + 1) \text{ and } |z| \geq R_4 \text{ where } R_1 < R_4.$$

By an argument analogous to that of the preceding section, we have

$$f(z) = -(1/2\pi i) \int_{C_1} \sum_{n=0}^{\infty} F_n(t) V_n(z) f(t)\, dt$$

where the integration is in the clockwise direction. If we take it in the counterclockwise direction we have

$$f(z) = (1/2\pi i) \int_{C_1} \sum_{n=0}^{\infty} F_n(t) V_n(z) f(t)\, dt.$$

Integrating term by term as is permissible we have

$$f(z) = \sum_{n=0}^{\infty} \alpha_n V_n(z), \text{where } \alpha_n = (1/2\pi i) \int_{C_1} F_n(t) f(t)\, dt,$$

from which the theorem follows.

7. *Analogue of the Laurent Expansions.* Let C_1, C_2, C_3, C_4 be, respectively, the circles $|z| = R_1$, $|z| = R_2$, $|z| = R_3$, $|z| = R_4$ where $R_1 < R_2 < R_3 < R_4$ and $R_3 < N/(M_N + 1)$. Let $f(z)$ be a single-valued, analytic function of z within and upon the boundary of the ring bounded by C_1C_4. If z is a point within or upon the boundary of the ring bounded by C_2C_3 we have by means of Cauchy's integral formula

$$f(z) = (1/2\pi i) \int_{C_4} \frac{f(t)}{t - z}\, dt + (1/2\pi i) \int_{C_1} \frac{f(t)}{z - t}\, dt$$

where both integrations are performed in the counterclockwise direction. With the aid of theorem I we obtain

$$f(z) = (1/2\pi i) \int_{C_4} [\sum_{n=0}^{\infty} V_n(t) F_n(z)] f(t)\, dt + (1/2\pi i) \times \int_{C_1} [\sum_{n=0}^{\infty} F_n(t) V_n(z)] f(t)\, dt.$$

We may integrate term by term so that

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z) + \sum_{n=0}^{\infty} a_n' V_n(z),$$

where $a_n = (1/2\pi i) \int_{C_4} V_n(t) f(t)\, dt$ and $a_n' = (1/2\pi i) \int_{C_1} F_n(t) f(t)\, dt.$

This expansion is valid and converges uniformly with respect to z if z is within or upon the boundary of the ring C_2C_3.

THEOREM IV. *Let C_1, C_2, C_3, C_4 be, respectively, the circles $|z| = R_1$, $|z| = R_2$, $|z| = R_3$, $|z| = R_4$ where $R_1 < R_2 < R_3 < R_4$ and $R_3 < N/(M_N + 1)$. If $f(z)$ is a single-valued, analytic function of z within and upon the boundary of the ring C_1C_4, then the expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z) + \sum_{n=0}^{\infty} a_n' V_n(z),$$

where $a_n = (1/2\pi i) \int_{C_4} V_n(t) f(t)\, dt$ and $a_n' = (1/2\pi i) \int_{C_1} F_n(t) f(t)\, dt,$

is valid and converges uniformly with respect to z when z is within or upon the boundary of the ring C_2C_3.

8. *Relations to the Taylor and Laurent Expansions.* If $f(z)$ is analytic in the neighborhood of the origin let its Taylor expansion be

$$f(z) = \sum_{s=0}^{\infty} b_s z^s.$$

We substitute this expansion in the expression for the coefficients of the expansion in theorem II and also replace $V_n(t)$ by its value. We have

$$a_n = (1/2\pi i) \int_{C_4} V_n(t) f(t)\, dt$$

$$= (1/2\pi i) \int_{C_4} (\sum_{r=0}^{n} \gamma_{rn}/t^{r+1}) \ (\sum_{s=0}^{\infty} b_s t^s) dt$$

$$= (1/2\pi i) \int_{C_4} \sum_{r=0}^{n} \sum_{s=0}^{\infty} \gamma_{rn} b_s t^{s-r-1} dt.$$

Hence, by performing the integration,

$$a_n = \sum_{s=0}^{n} b_s \gamma_{sn}.$$

If $f(z)$ is analytic in the neighborhood of infinity its Taylor expansion takes the form

$$f(z) = \sum_{r=0}^{\infty} \beta_r z^{-r}.$$

We substitute in the expression for the coefficients in theorem III thus,

$$\alpha_n = (1/2\pi i) \int_{C_1} F_n(t) f(t) dt$$

$$= (1/2\pi i) \int_{C_1} (\sum_{s=0}^{\infty} c_{ns} t^{n+s}) \ (\sum_{r=0}^{\infty} \beta_r z^{-r}) dt$$

$$= (1/2\pi i) \int_{C_1} \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} c_{ns} \beta_r t^{n+s-r} dt.$$

Hence

$$\alpha_n = \sum_{s=0}^{\infty} c_{ns} \beta_{n+s+1}.$$

If $f(z)$ has the Laurent expansion

$$f(z) = \sum_{s=0}^{\infty} b_s z^s + \sum_{s=1}^{\infty} b_s' z^{-s}$$

valid in the ring C_1C_4 of theorem IV then the coefficients of the expansion of that theorem are related to these coefficients thus,

$$a_n = \sum_{s=0}^{n} b_s \gamma_{sn},$$

$$a_n' = \sum_{s=0}^{\infty} c_{ns} b'_{n+s}.$$

These relations may be useful in obtaining the expansions of theorems II, III, IV provided the Taylor and Laurent expansions are readily obtainable.

9. *Orthogonality Properties.* We shall establish the following:

$$\int_C F_n(z)F_m(z)\,dz = 0, \quad (m = n \text{ and } m \neq n),$$

where C is any closed contour in the region of analyticity of both $F_n(z)$ and $F_m(z)$.

$$\int_C V_n(z)V_m(z)\,dz = 0, \quad (m = n \text{ and } m \neq n),$$

where C is any closed contour in the finite plane not passing through the origin.

$$\begin{aligned}\int_C V_n(z)F_n(z)\,dz &= 2\pi i k, \ (C \text{ enclosing and not passing through the origin}),\\ &= 0, \ (C \text{ not enclosing nor passing through the origin}),\end{aligned}$$

where C is a closed contour in the region of analyticity of $F_n(z)$, and k is the excess of the number of positive circuits over the number of negative circuits made by the contour about the origin.

$$\int_C V_n(z)F_m(z)\,dz = 0, \quad (m \neq n),$$

where C is any closed contour in the region of analyticity of $F_m(z)$ but not passing through the origin if $m < n$.

The value of the first integral follows at once from Cauchy's theorem. The product $V_n(z)\ V_m(z)$ is a polynomial in $1/z$ with $(1/z)^2$ as the lowest power. We have therefore the value of the second integral. The value of the third integral is obtained at once by the residue of the integrand at zero. If $m > n$ the product $V_n(z)\ F_m(z)$ is a power series in z so the value of the fourth integral is zero in that case. If $m < n$ we have from (1) and the expression for $V_n(z)$ in theorem II

$$\int_C V_n(z)F_m(z)\,dz = \int_C \Big(\sum_{r=0}^{n}\gamma_{rn}/z^{r+1}\Big)\Big(\sum_{s=0}^{\infty}c_{ms}z^{m+s}\Big)dz = 2\pi i\sum_{s=0}^{n-m}\gamma_{m+s,n}c_{ms}.$$

Writing $m + \nu$ for n the last summation becomes

$$\sum_{s=0}^{\nu}\gamma_{m+s,m+\nu}c_{ms}.$$

We then substitute for $\gamma_{m+s,m+\nu}$ from (4). When the determinant for $\gamma_{m,m+\nu}$ is expanded according to the elements of the first column and the result compared with the foregoing sum, the latter is seen to vanish.

The expansions of theorems II, III and IV are known to exist in some small regions under the conditions set forth in the theorems. The integral formulae just obtained enable us to readily determine the coefficients of the expansions. For example, the expansion of theorem II exists and converges uniformly with respect to z within and upon some small circle described about the origin. We may then multiply this expansion by $V_m(z)$ and integrate term by term in the right member. The integrations are in the positive direction. Also C lies in the circle of existence of the expansion and encloses the origin in a single circuit. We obtain

$$\int_C V_m(z)f(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C V_m(z)F_n(z)\,dz.$$

By the last two integral formulae

$$\int_C V_m(z)f(z)\,dz = 2\pi i\, a_m;$$

this determines a_m.

10. *Results in the neighborhood of $z = a$.* By replacing z by $(z-a)$ and t by $(t-a)$ in the result stated in theorem I we obtain

$$1/[(t-a)-(z-a)] = \sum_{n=0}^{\infty} V_n(t-a)F_n(z-a).$$

It is readily seen that this expansion has the same properties in the neighborhood of a as are possessed by the expansion in theorem I in the neighborhood of the origin. By means of this result and Cauchy's integral formula we are able to expand any function $f(z)$, analytic and single-valued within and upon the boundary of a closed region bounded by C_4, the equation of C_4 being $|z-a| = R_4$. We have

$$f(z) = \frac{1}{2\pi i}\int_{C_4} \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\int_{C_4} \frac{f(t)}{(t-a)-(z-a)}\,dt$$

$$= (1/2\pi i)\int_{C_4} \sum_{n=0}^{\infty} V_n(t-a)F_n(z-a)f(t)\,dt$$

$$= \sum_{n=0}^{\infty} \bar{a}_n F_n(z-a), \text{ where } \bar{a}_n = (1/2\pi i)\int_{C_4} V_n(t-a)f(t)\,dt.$$

This last expansion is valid and converges uniformly with respect to z if $|z-a| \leq R_1 < R_4$ and $R_1 < N/(M_N+1)$.

THEOREM V. *If $f(z)$ be a single-valued, analytic function of z within and upon C_4 whose equation is $|z-a|=R_4$, then the expansion*

$$f(z)=\sum_{n=0}^{\infty}\bar{a}_nF_n(z-a),\ \textit{where}\ \bar{a}_n=(1/2\pi i)\int_{C_4}V_n(t-a)f(t)\,dt,$$

is valid and converges uniformly with respect to z if z is within or upon the circle $|z-a|=R_1$ where $R_1<N/(M_N+1)$ and $R_1<R_4$.

It is clear that all the results of § 6 to § 9 inclusive could be established for the neighborhood of $z=a$ as well as for the neighborhood of $z=0$.

11. *Expansion of Functions of Several Variables.* The expansion theory obtained in the preceding sections can be easily extended for the expansion of functions of several variables in multiple series. As an illustration, an expansion similar to that of theorem II will be carried through for a function of two independent variables. From this special case the generalization to the case involving n variables will be readily seen.

We take two general sequences of functions of z_1 and z_2, respectively,

$$F_m^{(p)}(z_p)=\sum_{s=0}^{\infty}c_{ms}^{(p)}z_p^{m+s},\quad c_{m0}^{(p)}=1,\quad m=0,1,2,\cdots,\quad p=1,2,$$

similar to the sequence defined in § 2 and place upon them similar hypotheses. The power series defining each function of the sequence $F_m^{(p)}(z_p)$ is assumed to be convergent within some circle of radius $R^{(p)}$ described about the origin as a center where $R^{(p)}$ is independent of m. By the theorem quoted in § 2 there is a greatest value $M_N^{(p)}(m)$ which $|F_m^{(p)}(z_p)|$ takes on on a circle of radius $N^{(p)}<R^{(p)}$ described about the origin and we have

$$M_N^{(p)}(m)\geq|c_{ms}^{(p)}|(N^{(p)})^{m+s}.$$

As before we then make the further assumption that there exists a finite $M_N^{(p)}$ such that for all values of m we have

$$M_N^{(p)}\geq M_N^{(p)}(m)(N^{(p)})^{-m},\quad p=1,2.$$

The results of §§ 3, 4 follow as before for each sequence $F_m^{(p)}(z_p)$, $m=0,1,2,\cdots$, $p=1,2$.

Let z_1, t_1 and z_2, t_2 be two pairs of complex variables in the z_1 and z_2 planes, respectively. Let them be restricted as follows:

$$\begin{cases}|z_1|\leq R_1^{(1)}<N^{(1)}/(M_N^{(1)}+1), & |t_1|\geq R_4^{(1)},\quad R_1^{(1)}<R_4^{(1)};\\ |z_2|\leq R_1^{(2)}<N^{(2)}/(M_N^{(2)}+1), & |t_2|\geqq R_4^{(2)},\quad R_1^{(2)}<R_4^{(2)}.\end{cases}$$

Also we take

$$1/(t_1-z_1)=\sum_{m_1=0}^{\infty} V_{m_1}^{(1)}(t_1)F_{m_1}^{(1)}(z_1), \quad 1/(t_2-z_2)=\sum_{m_2=0}^{\infty} V_{m_2}^{(2)}(t_2)F_{m_2}^{(2)}(z_2),$$

where $V_{m_1}^{(1)}(t_1)$ and $V_{m_2}^{(2)}(t_2)$ are associated with $F_{m_1}^{(1)}(z_1)$ and $F_{m_2}^{(2)}(z_2)$, respectively, in the same manner as $V_m(t)$ is associated with $F_m(z)$. Allow t_1 and t_2 to be variable points outside or upon $C_4^{(1)}$ and $C_4^{(2)}$, respectively, where $C_4^{(1)}$ is the circle $|z_1|=R_4^{(1)}$ and $C_4^{(2)}$ the circle $|z_2|=R_4^{(2)}$, and z_1 and z_2 variable points, respectively, within or upon $C_1^{(1)}$ defined by $|z_1|=R_1^{(1)}$ and $C_1^{(2)}$ defined by $|z_2|=R_1^{(2)}$. Then the expansions for $1/(t_1-z_1)$ and $1/(t_2-z_2)$ converge uniformly with respect to t_1, z_1 and t_2, z_2, respectively. Also these expansions converge absolutely. On multiplying them together term by term we obtain the equation

$$1/(t_1-z_1)\cdot 1/(t_2-z_2)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} V_{m_1}^{(1)}(t_1)V_{m_2}^{(2)}(t_2)F_{m_1}^{(1)}(z_1)F_{m_2}^{(2)}(z_2),$$

where the double series is absolutely and uniformly convergent. Let $f(z_1,z_2)$ be a function of the two independent, complex variables z_1 and z_2. Let it be analytic and single-valued in z_1 when z_1 is within or upon the circle $C_4^{(1)}$ and z_2 is held fixed within or upon the circle $C_4^{(2)}$. Likewise let it be analytic and single-valued in z_2 when z_2 is within or upon $C_4^{(2)}$ and z_1 is held fixed within or upon $C_4^{(1)}$. By Cauchy's generalized integral formula we have

$$f(z_1,z_2)=\frac{1}{(2\pi i)^2}\int_{C_4^{(1)}}\int_{C_4^{(2)}}\frac{f(t_1,t_2)}{(t_1-z_1)(t_2-z_2)}dt_2dt_1$$

$$=1/(2\pi i)^2\int_{C_4^{(1)}}\int_{C_4^{(2)}}\Big[\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}V_{m_1}^{(1)}(t_1)V_{m_2}^{(2)}(t_2)F_{m_1}^{(1)}(z_1)F_{m_2}^{(2)}(z_2)\Big]$$
$$\times[f(t_1,t_2)/(t_1-z_1)(t_2-z_2)]dt_2dt_1.$$

Hence

$$f(z_1,z_2)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}A_{m_1m_2}F_{m_1}^{(1)}(z_1)F_{m_2}^{(2)}(z_2),$$

where $\quad A_{m_1m_2}=1/(2\pi i)^2\int_{C_4^{(1)}}\int_{C_4^{(2)}}f(t_1,t_2)V_{m_1}^{(1)}(t_1)V_{m_2}^{(2)}(t_2)dt_2dt_1.$

This expansion is valid and converges uniformly with respect to z_1, z_2 when z_1, z_2 are, respectively, within or upon the circles $|z_1|=R_1^{(1)}$, $|z_2|=R_1^{(2)}$. Since the extension of this result to any number of variables is obvious, we shall state the following theorem for n variables:

THEOREM VI. *Let $f(z_1, z_2, \cdots, z_n)$ be a function of the independent, complex variables $z_1, z_2, \cdots, z_n$. Designate the circles $|z_1| = R_4^{(1)}$, $|z_2| = R_4^{(2)}, \cdots, |z_n| = R_4^{(n)}$ by $C_4^{(1)}, C_4^{(2)}, \cdots, C_4^{(n)}$, respectively. Let $f(z_1, z_2, \cdots, z_n)$ be analytic and single-valued in z_1 when z_1 is within or upon $C_4^{(1)}$ and $z_2, z_3, \cdots, z_n$ are held fixed within or upon $C_4^{(2)}, C_4^{(3)} \cdots$, $C_4^{(n)}$, respectively. Let it be similarly analytic and single-valued in $z_2, z_3, \cdots, z_n$. Then the expansion*

$$f(z_1, z_2, \cdots, z_n)$$
$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} A_{m_1 m_2 \ldots m_n} F_{m_1}^{(1)}(z_1) F_{m_2}^{(2)}(z_2) \cdots F_{m_n}^{(n)}(z_n),$$

where

$$A_{m_1 m_2 \ldots m_n} = 1/(2\pi i)^n \int_{C_4^{(1)}} \int_{C_4^{(2)}} \cdots$$
$$\int_{C_4^{(n)}} f(t_1, t_2 \cdots t_n) \; V_{m_1}^{(1)}(t_1) V_{m_2}^{(2)}(t_2) \cdots V_{m_n}^{(n)}(t_n) \, dt_n \cdots dt_2 dt_1,$$

is valid and converges uniformly with respect to $z_1, z_2, \cdots, z_n$ when $z_1, z_2, \cdots, z_n$ are, respectively, within or upon the circles $|z_1| = R_1^{(1)}$, $|z_2| = R_1^{(2)}$, $\cdots$, $|z_n| = R_1^{(n)}$ where $R_1^{(1)} < R_4^{(1)}$, $R_1^{(2)} < R_4^{(2)}$, $\cdots$, $R_1^{(n)} < R_4^{(n)}$ and $R_1^{(1)} < N^{(1)}/(M_N^{(1)} + 1)$, $R_1^{(2)} < N^{(2)}/(M_N^{(2)} + 1), \cdots$, $R_1^{(n)} < N^{(n)}/(M_N^{(n)} + 1)$.

12. *Further Results for Functions of Several Variables.* Results analogous to those obtained for analytic functions of a single variable in § 6 to § 10 inclusive can be obtained for functions of two independent variables or, in fact, of n independent variables. We shall not attempt to establish these results in detail. However, as an example, we shall work out the analogue of the Laurent expansion for the case of a function of two independent variables.

Let z_1, t_1 and z_2, t_2 be two pairs of independent, complex variables in the z_1 and z_2 planes, respectively. Let $C_1^{(1)}, C_2^{(1)} C_3^{(1)}, C_4^{(1)}$ be four circles in the z_1 plane concentric at the origin and with radii, respectively, $R_1^{(1)}, R_2^{(1)}$, $R_3^{(1)}, R_4^{(1)}$ where $R_1^{(1)} < R_2^{(1)} < R_3^{(1)} < R_4^{(1)}$. Also $C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}$ be similarly defined in the z_2 plane. Let $f(z_1, z_2)$ be analytic and single-valued in z_1 when z_1 is within or upon the ring $C_1^{(1)} C_4^{(1)}$ and z_2 is held fixed within or upon the ring $C_1^{(2)} C_4^{(2)}$. Likewise let it be analytic and single-valued in z_2 when z_2 is within or upon $C_1^{(2)} C_4^{(2)}$ and z_1 is held fixed within or upon $C_1^{(1)} C_4^{(1)}$. Further let the R's be restricted by the relations

$$R_3^{(1)} < N^{(1)}/(M_N^{(1)} + 1), \; R_3^{(2)} < N^{(2)}/(M_N^{(2)} + 1).$$

As in § 11 we have the expansions

$$\begin{cases} 1/(t_1 - z_1) = \sum_{m_1=0}^{\infty} V_{m_1}{}^{(1)}(t_1) F_{m_1}{}^{(1)}(z_1), \\ 1/(t_2 - z_2) = \sum_{m_2=0}^{\infty} V_{m_2}{}^{(2)}(t_2) F_{m_2}{}^{(2)}(z_2), \end{cases}$$

which are uniformly convergent with respect to t_1, z_1 and t_2, z_2 if t_1, t_2 are outside or upon $C_4{}^{(1)}$, $C_4{}^{(2)}$, respectively, and z_1, z_2 are within or upon $C_3{}^{(1)}$, $C_3{}^{(2)}$, respectively. Also we have the expansions

$$\begin{cases} 1/(z_1 - t_1) = \sum_{m_1=0}^{\infty} V_{m_1}{}^{(1)}(z_1) F_{m_1}{}^{(1)}(t_1), \\ 1/(z_2 - t_2) = \sum_{m_2=0}^{\infty} V_{m_2}{}^{(2)}(z_2) F_{m_2}{}^{(2)}(t_2), \end{cases}$$

which are uniformly convergent with respect to t_1, z_1 and t_2, z_2 if t_1, t_2 are within or upon $C_1{}^{(1)}$, $C_1{}^{(2)}$, respectively, and z_1, z_2 are outside or upon $C_2{}^{(1)}$, $C_2{}^{(2)}$, respectively. These expansions converge absolutely. We treat the results of multiplying them together term by term as in § 11. Then by Cauchy's generalized integral formula we have (all integrations being in the counterclockwise direction)

$$f(z_1, z_2) = \frac{1}{2\pi i}\int_{C_4{}^{(1)}} \frac{f(t_1, z_2)}{(t_1 - z_1)} dt_1 + \frac{1}{2\pi i}\int_{C_1{}^{(1)}} \frac{f(t_1, z_2)}{(z_1 - t_1)} dt_1$$

and

$$f(t_1, z_2) = \frac{1}{2\pi i}\int_{C_4{}^{(2)}} \frac{f(t_1, t_2)}{(t_2 - z_2)} dt_2 + \frac{1}{2\pi i}\int_{C_1{}^{(2)}} \frac{f(t_1, t_2)}{(z_2 - t_2)} dt_2.$$

Substituting from the second in the first we have

$$\begin{aligned} f(z_1, z_2) = {} & \frac{1}{(2\pi i)^2}\int_{C_4{}^{(1)}}\int_{C_4{}^{(2)}} \frac{f(t_1, t_2)}{(t_2 - z_2)(t_1 - z_1)} dt_2 dt_1 \\ & + \frac{1}{(2\pi i)^2}\int_{C_4{}^{(1)}}\int_{C_1{}^{(2)}} \frac{f(t_1, t_2)}{(z_2 - t_2)(t_1 - z_1)} dt_2 dt_1 \\ & + \frac{1}{(2\pi i)^2}\int_{C_1{}^{(1)}}\int_{C_4{}^{(2)}} \frac{f(t_1, t_2)}{(t_2 - z_2)(z_1 - t_1)} dt_2 dt_1 \\ & + \frac{1}{(2\pi i)^2}\int_{C_1{}^{(1)}}\int_{C_1{}^{(2)}} \frac{f(t_1, t_2)}{(z_2 - t_2)(z_1 - t_1)} dt_2 dt_1. \end{aligned}$$

Using the expansions just obtained we have

$$f(z_1, z_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(1)}{}_{m_1m_2} F_{m_1}{}^{(1)}(z_1) F_{m_2}{}^{(2)}(z_2)$$
$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(2)}{}_{m_1m_2} F_{m_1}{}^{(1)}(z_1) V^{(2)}{}_{m_2}(z_2)$$
$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(3)}{}_{m_1m_2} V_{m_1}{}^{(1)}(z_1) F_{m_2}{}^{(2)}(z_2)$$
$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(4)}{}_{m_1m_2} V_{m_1}{}^{(1)}(z_1) V_{m_2}{}^{(2)}(z_2),$$

where

$$A^{(1)}{}_{m_1m_2} = \frac{1}{(2\pi i)^2} \int_{C_4^{(1)}} \int_{C_4^{(2)}} V_{m_1}{}^{(1)}(t_1) V_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1,$$

$$A^{(2)}{}_{m_1m_2} = \frac{1}{(2\pi i)^2} \int_{C_4^{(1)}} \int_{C_1^{(2)}} V_{m_1}{}^{(1)}(t_1) F_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1,$$

$$A^{(3)}{}_{m_1m_2} = \frac{1}{(2\pi i)^2} \int_{C_1^{(1)}} \int_{C_4^{(2)}} F_{m_1}{}^{(1)}(t_1) V_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1,$$

$$A^{(4)}{}_{m_1m_2} = \frac{1}{(2\pi i)^2} \int_{C_1^{(1)}} \int_{C_1^{(2)}} F_{m_1}{}^{(1)}(t_1) F_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1.$$

II. Expansions in Confluent Hypergeometric Functions.

13. *Certain Confluent Hypergeometric Functions Satisfying the Hypotheses in* § 2. We define $M_{km}(z)$ as follows,

$$M_{km}(z) = z^{\frac{1}{2}+m} e^{-z/2}$$
$$\times \left\{ 1 + \sum_{r=1}^{\infty} \frac{(\tfrac{1}{2}+m-k)(\tfrac{3}{2}+m-k)\cdots(r-\tfrac{1}{2}+m-k)}{(2m+1)(2m+2)\cdots(2m+r)} \frac{z^r}{r!} \right\},$$

where $2m$ is not a negative integer. Then $M_{km}(z)$ is an integral of

$$\frac{d^2y}{dz^2} + \left\{ -\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4}-m^2}{z^2} \right\} y = 0. \tag{11}$$

This equation is obtained from the confluent hypergeometric equation

$$\frac{d^2u}{dz^2} + \frac{du}{dz} + \left\{ \frac{k}{z} - \frac{\tfrac{1}{4}-m^2}{z^2} \right\} u = 0$$

by means of the transformation $u = e^{-z/2}y$. The series for $M_{km}(z)$ converges for all finite values of z*. We shall write

$$\overline{M}_{km}(z) = z^{-\frac{1}{2}}e^{z/2}M_{km}(z)$$

and deal with expansions in terms of $\overline{M}_{km}(z)$ allowing m to range over zero and the positive integers.

For the circle of radius R mentioned in § 2 we may take any circle described about the origin. Then, using $M_N(m)$, N and M_N in the same senses as in § 2 we have

$$M_N(m) \leq N^m$$

$$\times \left\{ 1 + \sum_{r=1}^{\infty} \frac{|\tfrac{1}{2}+m-k|\ |\tfrac{3}{2}+m-k| \cdots |r-\tfrac{1}{2}+m-k|}{(2m+1)\ (2m+2) \cdots (2m+r)} \frac{N^r}{r!} \right\}.$$

The rth term in the summation on the right has r factors of the form

$$|\alpha - \tfrac{1}{2} + m - k|/(2m+\alpha), \quad \alpha = 1, 2, \cdots, r.$$

Now

$$|\alpha - \tfrac{1}{2} + m - k|/(2m+\alpha) \leq (m + \alpha + |k - \tfrac{1}{2}|)/(m + \alpha + m).$$

Hence there exists a constant L such that

$$(12) \qquad L \geq |\alpha - \tfrac{1}{2} + m - k|/(2m+\alpha)$$

independently of m and α and for any fixed k. We then have

$$M_N(m) \leq N^m \{1 + \sum_{r=1}^{\infty} (LN)^r/r!\} = N^m e^{LN},$$

and M_N may be taken

$$M_N = e^{LN}.$$

The sequence of functions $\overline{M}_{km}(z)$, $m = 0, 1, 2, \cdots$ then satisfies the hypotheses of § 2 and the theorems of the first division apply to it. The expansions there obtained converge in some small regions. We now proceed to extend those regions by further investigation.

14. *Expansion of* $1/(t-z)$. Now we have an expansion of the form

$$z^n = \sum_{\nu=0}^{\infty} c_{n+\nu}\, \overline{M}_{k,n+\nu}(z).$$

*** See Whittaker and Watson, *A Course of Modern Analysis* (3d Ed., Cambridge, 1920), Chapter XVI, for discussion of solutions of (11) and further references.**

By inserting in the right member the expansion of $\overline{M}_{km}(z)$ and equating the coefficients of like powers of z, we find that the c's satisfy the recurrence relations

$$\sum_{m=0}^{r-1} \frac{(\tfrac{1}{2}+m-k)(\tfrac{3}{2}+m-k)\cdots(r-\tfrac{1}{2}-k)}{(2m+1)(2m+2)\cdots(m+r)\ (r-m)!}\, c_m + c_r = 1,\ r = n,$$
$$= 0,\ r \neq n;$$

from which it follows that

$$c_n = 1 \text{ and } c_{n+\nu} = (-)^\nu \frac{(n+\tfrac{1}{2}-k)(n+\tfrac{3}{2}-k)\cdots(n+\nu-\tfrac{1}{2}-k)}{(2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)\ \nu!}.$$

Making use of (12) we have

$$|\,\overline{M}_{km}(z)\,| \leqq |\,z\,|^m e^{L|z|}.$$

The expansion for z^n is then dominated by the series

$$e^{L|z|}\,|\,z\,|^n + \sum_{\nu=1}^{\infty} \left| \frac{(n+\tfrac{1}{2}-k)(n+\tfrac{3}{2}-k)\cdots(n+\nu-\tfrac{1}{2}-k)}{(2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)\ \nu!} \right|$$
$$\times\ |\,z\,|^{n+\nu} e^{L|z|}.$$

By the ratio test it is seen that this series converges for all finite values of z and the same is true of the series dominated. It follows that the series for z^n is absolutely and uniformly convergent in any finite closed region. Therefore since the series represents z^n in the neighborhood of zero it represents z^n everywhere. Now from the equation

$$1/(t-z) = \sum_{n=0}^{\infty} z^n/t^{n+1}, \quad |\,z\,| < |\,t\,|,$$

we have

$$(13) \qquad 1/(t-z) = \sum_{n=0}^{\infty} 1/t^{n+1} \{\overline{M}_{kn}(z)$$
$$+ \sum_{\nu=1}^{\infty} (-)^\nu \frac{(n+\tfrac{1}{2}-k)(n+\tfrac{3}{2}-k)\cdots(n+\nu-\tfrac{1}{2}-k)}{\nu!\ (2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)} \overline{M}_{k,n+\nu}(z)\}.$$

This series is dominated by

$$\sum_{n=0}^{\infty} e^{L|z|}/|\,t\,|^{n+1} \{\,|\,z\,|^n$$
$$+ \sum_{\nu=1}^{\infty} \frac{|\,n+\tfrac{1}{2}-k\,|\ |\,n+\tfrac{3}{2}-k\,| \cdots |\,n+\nu-\tfrac{1}{2}-k\,|}{\nu!\ (2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)}\, |\,z\,|^{n+\nu}\}.$$

By an argument similar to that used in establishing (12) we have

$$|\,n+\alpha-\tfrac{1}{2}-k\,|/(2n+2\alpha-1) \leqq L', \quad \alpha = 1, 2, \cdots, \nu;$$

and

$$(14)\quad \sum_{n=0}^{\infty} e^{L|z|}/|\,t\,|^{n+1}\ |\,z\,|^{n} \sum_{\nu=0}^{\infty} (L'|\,z\,|)^{\nu}/\nu! = e^{(L+L')|z|} \sum_{n=0}^{\infty} |\,z\,|^{n}/|\,t\,|^{n+1}$$

which is convergent if $|\,z\,| < |\,t\,|$ and consequently (13) is absolutely convergent if $|\,z\,| < |\,t\,|$. By an argument similar to that of § 4 we may rearrange (13) obtaining

$$1/(t-z) = \sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z), \text{ where } V_{k0}(t) = 1/t$$

$$\text{and } V_{kn}(t) = \sum_{r=0}^{n-1} (-)^{n+r} \frac{(r+\tfrac{1}{2}-k)(r+\tfrac{3}{2}-k)\cdots(n-\tfrac{1}{2}-k)}{(n-r)!\,(n+r)(n+r+1)\cdots(2n-1)t^{r+1}} + \frac{1}{t^{n+1}}, \ n \geqq 1.$$

We may state the following:

If z and t be two complex variables such that $|\,z\,| \leq R_1$ and $|\,t\,| \geq R_4$ where $R_1 < R_4$ then the expansion

$$(15)\qquad 1/(t-z) = \sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z), \text{ where } V_{k0} = 1/t$$

$$\text{and}\quad \mathrm{V}_{kn}(t) = \sum_{r=0}^{n-1} (-)^{n+r} \frac{(r+\tfrac{1}{2}-k)(r+\tfrac{3}{2}-k)\cdots(n-\tfrac{1}{2}+k)}{(n-r)!\,(n+r)(n+r+1)\cdots(2n-1)t^{r+1}} + \frac{1}{t^{n+1}}, \quad n \geq 1,$$

is valid and converges uniformly with respect to t and z.

15. *The differential Equation Satisfied by $V_{kn}(t)$.* By means of the transformation $M_{kn}(z) = e^{-z/2}z^{\frac{1}{2}}\overline{M}_{kn}(z)$ applied to the differential equation (11) we find that $\overline{M}_{kn}(z)$ satisfies the equation

$$z^2 \frac{d^2u}{dz^2} + (z-z^2)\frac{du}{dz} + \{(k-\tfrac{1}{2})z - n^2\}u = 0.$$

We differentiate both members of the equation (15) with respect to z and multiply through by the corresponding coefficients from the above differential equation. Then using that equation we obtain

$$2z^2/(t-z)^3 + (z-z^2)/(t-z)^2 + (k-\tfrac{1}{2})/(t-z)$$
$$= \sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z)\cdot n^2.$$

Expanding the numerators of the terms of the left member in powers of $(t-z)$ and simplifying, we have

$$2t^2/(t-z)^3-(3t+t^2)/(t-z)^2+[(k+3/2)t+1]/(t-z)-k-1/2$$
$$=\sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z)\cdot n^2.$$

Again differentiating both members of the equation (15), this time with respect to t, and making use of the last equation we find a differential equation satisfied by $V_{kn}(t)$ to be

$$t^2\frac{d^2y}{dt^2}+(t^2+3t)\frac{dy}{dt}+[(k+3/2)t+(1-n^2)]y$$
$$=(1/2+k)\frac{(1/2-k)(3/2-k)\cdots(n-1/2-k)}{n!\,n(n+1)\cdots(2n-1)}.$$

16. *Expansions in Terms of $V_{kn}(z)$ and $\overline{M}_{kn}(z)$.* By an argument similar to that in §§ 5, 6, 7 we can establish the following theorems:

THEOREM VII. *If $f(z)$ be a single-valued, analytic function of z within and upon C_4, the equation of C_4 being $|z|=R_4$, then the expansion*

$$f(z)=\sum_{n=0}^{\infty} a_{kn}\overline{M}_{kn}(z),\ \text{where}\ a_{kn}=(1/2\pi i)\int_{C_4} V_{kn}(t)f(t)\,dt,$$

is valid and converges uniformly with respect to z if z is within or upon the circle $|z|=R_1$, where $R_1<R_4$.

COROLLARY. *If $f(z)$ is analytic throughout the finite plane then the foregoing expansion for $f(z)$ is valid throughout the finite plane.*

THEOREM VIII. *If $f(z)$ be a single-valued, analytic function of z upon and outside C_1, the equation of C_1 being $|z|=R_1$, then the expansion*

$$f(z)=\sum_{n=0}^{\infty} \alpha_{kn}V_{kn}(z),\ \text{where}\ \alpha_{kn}=(1/2\pi i)\int_{C_1}\overline{M}_{kn}(t)f(t)\,dt,$$

is valid and converges uniformly with respect to z if z is outside or upon the circle $|z|=R_4$ where $R_1<R_4$.

THEOREM IX. *Let C_1, C_2, C_3, C_4 be, respectively, the circles $|z|=R_1$,*

$|z| = R_2$, $|z| = R_3$, $|z| = R_4$ *where* $R_1 < R_2 < R_3 < R_4$. *If* $f(z)$ *be a single-valued, analytic function of* z *within and upon the boundary of the ring* C_1C_4 *then the expansion*

$$f(z) = \sum_{n=0}^{\infty} a_{kn}\overline{M}_{kn}(z) + \sum_{n=0}^{\infty} a'_{kn}V_{kn}(z),$$

where $a_{kn} = (1/2\pi i)\int_{C_4} V_{kn}(t)f(t)\,dt$ *and* $a'_{kn} = (1/2\pi i)\int_{C_1} \overline{M}_{kn}(t)f(t)\,dt$,

is valid and converges uniformly with respect to z *when* z *is within or upon the boundary of the ring* C_2C_3.

We can establish the relationships between the coefficients of these expansions and those of the Taylor and Laurent expansions of $f(z)$. Orthogonality properties for $V_{kn}(z)$ and $\overline{M}_{kn}(z)$ similar to those in § 9 can also be proven. Results may be obtained in the neighborhood of $z = a$. The theory may be extended to the expansions of functions of n variables in multiple series.

A Modern Presentation of Grassmann's Tensor Analysis.

By Helen Barton.

1. *Introduction.* Since Einstein advanced his Theory of Relativity in 1916, there has been a considerable increase in interest in what is known as Tensor Analysis. Mathematicians and physicists alike are realizing the power of this method of treatment; especially since it has led to the association of physical quantities which hitherto had seemed unrelated. The idea of a tensor, however, far antedates Einstein, it being introduced by Grassmann in 1844.*

In this work of Grassmann's *Die Ausdehnungslehre,* the author gives a rather exhaustive algebraic treatment of these quantities, which he termed "extensive Grösse," but his ideas were often abstruse and vague, so that the work has never been used nor appreciated very much by mathematicians. The main part of this paper will be devoted to the development of explicit expressions for these "extensive Grösse," and for various combinations of them. It is hoped by this more definite treatment of the subject that Grassmann's work will be simplified and thereby become of greater value to the mathematical world.

One method of defining a tensor is by means of its components and the transformation of these components under a change of coordinates. If we form n functions of n ordered numbers in the x space

$$\phi^1(x^1, \cdots, x^n), \ \phi^2(x^1, \cdots, x^n), \cdots, \quad \phi^n(x^1, \cdots, x^n),$$

arbitrary except that the ratios $\phi^1/x^1, \phi^2/x^2, \cdots, \phi^n/x^n$ are of the same physical dimensions, and then form another set of n functions of the corresponding y's in the y space,

$$\psi^1(y^1, \cdots, y^n), \ \psi^2(y^1, \cdots, y^n), \cdots, \quad \psi^n(y^1, \cdots, y^n)$$

which are defined by the equation

* H. Grassmann, *Die Ausdehnungslehre,* Werke, Bd. 1, Leipzig, 1896.

$$\psi^r = \phi^\alpha \, \partial y^r / \partial x^\alpha \, *$$

then these n functions $\phi^1, \phi^2, \cdots, \phi^n$ are the n components of a contravariant tensor of rank one as presented in the x space, and $\psi^1, \psi^2, \cdots, \psi^n$ are the n components of the same contravariant tensor as presented in the y space.

Similarly, if we start with n functions of the x coordinates, $\phi_1, \phi_2, \cdots, \phi_n$ which are now subjected to the condition that each product $\phi_r x^r$ is of the same physical dimensions, and then form the corresponding functions of the y coordinates $\psi_1, \psi_2, \cdots, \psi_n$ where $\psi_r = \phi_\alpha \, \partial x^\alpha / \partial y^r$ then $\phi_1, \phi_2, \cdots, \phi_n$ are the n components of a covariant tensor of rank one as presented in the x space, and $\psi_1, \psi_2, \cdots, \psi_n$ are the n components of the same covariant tensor as presented in the y space.

Other tensors of each of these types, as well as combinations of the two, but of higher rank are defined in a similar way; but without going into further detail, it is evident that tensors, as viewed from this angle, are composite quantities, and their components presented in one space bear a certain definite relation to the components in another space.

Grassmann had a very different point of view. He was not interested in the presentation of these "extensive Grösse" with reference to any system of cordinates, nor in their transformations, but his work dealt with the various operations which could be applied to these quantities. To him, an "extensive Grösse" was a quantity built up from simple quantities which have both direction and magnitude. These "extensive Grösse," therefore, are defined by their parts or components in certain given directions. It follows from the definition that two "Grösse" are equal if the corresponding components of the two are equal.

Grassmann made no reference as to whether these "extensive Grösse" were contravariantly or covariantly presented, and the reason for this is clear. In Euclidean space, where the coordinates are orthogonal, we have the fundamental relation

$$\begin{aligned}(ds)^2 &= (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2 \\ &= (dy^1)^2 + (dy^2)^2 + \cdots + (dy^n)^2.\end{aligned}$$

If linear transformations only are considered, they are of the type $x^r = c_\alpha{}^r y^\alpha$, $y^r = c_r{}^\alpha x^\alpha$ and there is no distinction between the covariant and the contravariant presentation of a tensor.

---

* A Greek letter in an expression plays the rôle of an umbral symbol, i. e. it indicates that the expression is a summation of terms, each formed by assigning the n numerical values in succession to the umbral symbol.

2. *Outer Product of Tensors.* We know from ordinary Vector Analysis that two vectors may be combined into a vector or outer product, to give a new vector of the same type but of rank two, and its components are given by a two row determinant. Thus the r_1r_2 component of the outer product of the vectors a and b is given by

$$[ab]^{r_1r_2} = \begin{vmatrix} a^{r_1} & a^{r_2} \\ b^{r_1} & b^{r_2} \end{vmatrix}$$

but this might also be written in the form

$$A^{r_1r_2} = [ab]^{r_1r_2} = \begin{bmatrix} r_1 & r_2 \\ \alpha_1 & \alpha_2 \end{bmatrix} a^{\alpha_1} b^{\alpha_2}.*$$

The symbol here used $\begin{bmatrix} r_1 & r_2 \\ s_1 & s_2 \end{bmatrix}$ is known as the *generalized Kronecker symbol.*

The symbol is, as its name indicates, a generalization of the ordinary Kronecker symbol and we shall represent it in the form $\begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ s_1 & s_2 & \cdots & s_m \end{bmatrix}$ where m is any integer $1, 2, \cdots, n$ and $r_1, r_2, \cdots, r_m$ and $s_1, s_2, \cdots, s_m$ may take independently any of the n integral values. By definition, the symbol has but three distinct values:

$$\begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ s_1 & s_2 & \cdots & s_m \end{bmatrix} = 0$$

if $r_1, r_2, \cdots, r_m$ and $s_1, s_2, \cdots, s_m$ are not arrangements of the same set of m distinct integers;

$$\begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ s_1 & s_2 & \cdots & s_m \end{bmatrix} = \pm 1$$

according as it takes an even or odd number of interchanges to bring $r_1, r_2, \cdots, r_m$ into the same arrangement as $s_1, s_2, \cdots s_m$.

It should be noted further that this symbol is itself a tensor, mixed in type—covariant of rank m and contravariant of rank m.† Furthermore, this mixed tensor is non-metric, that is, it holds for all space, in which a point is merely a set of n ordered numbers.

If, however, we modify the symbol by fixing the order of one of the sets of labels, for example, $\begin{bmatrix} 1, & 2, & \cdots, & n \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}$, as we do in defining the

* F. D. Murnaghan, *American Mathematical Monthly*, Vol. 32 (1925), pp. 233-241.

† F. D. Murnaghan, *Bulletin of the American Mathematical Society*, Vol. 31 (1925), pp. 323-329.

complement of a tensor, it should be noted that it is no longer a mixed tensor of rank $2m$, but if it is multiplied by $\pm g^{1/2}$ it becomes a covariant tensor of rank m, where g is a determinant whose elements are the components of a covariant tensor of rank 2,—obtained from the metrical form

$$(ds)^2 = g_{\alpha\beta} x^\alpha x^\beta.$$

In normal space, where $g_{rs} = 0$ if $r \neq s$ and $g_{rr} = 1$, the determinant $g = 1$. Hence, since Grassmann was dealing, in effect, with points which were expressed in normal coordinates, the factor $g^{1/2}$ does not appear.

This symbol therefore lends itself to the formation of an outer product of any number of these vectors or tensors of rank one, thereby producing a tensor of higher rank,—equal to the number of these vectors contained in the product. But we need not restrict ourselves to the outer product of tensors of rank one. We may form the outer product of several tensors, each of different rank, and so obtain a tensor of still higher rank.

Thus if A is a tensor of rank p and B is a tensor of the same type as A and of rank q, and $p + q \leqq n$ (where n is the dimension of the space), then $[AB]^{r_1 \cdots r_p s_1 \cdots s_q}$ may be defined as

$$C^{r_1 \cdots r_p s_1 \cdots s_q} = \begin{bmatrix} r_1 r_2 \cdots r_p & s_1 s_2 \cdots s_q \\ \alpha_1 \alpha_2 \cdots \alpha_p & \beta_1 \beta_2 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

This is the explicit expression for the $r_1 \cdots r_p s_1 \cdots s_q$ component of the outer product of A and B, and it is readily seen that in this case C is a tensor of rank $p + q$.

Furthermore, it should be noted that C, the tensor obtained from the outer product of two or more tensors is alternating. For

$$[AB]^{r_1 r_2 \cdots r_p s_1 \cdots s_q} = \begin{bmatrix} r_1 r_2 \cdots r_p \, s_1 \cdots s_q \\ \alpha_1 \alpha_2 \cdots \alpha_p \, \beta_1 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

$$[AB]^{r_2 r_1 \cdots r_p s_1 \cdots s_q} = \begin{bmatrix} r_2 r_1 \cdots r_p \, s_1 \cdots s_q \\ \alpha_1 \alpha_2 \cdots \alpha_p \, \beta_1 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

It is evident that all odd permutations in the first case will be even in the second, and vice versa; hence, from the definition of the generalized Kronecker symbol,

$$[AB]^{r_1 r_2 \cdots r_p s_1 \cdots s_q} = -[AB]^{r_2 r_1 \cdots r_p s_1 \cdots s_q}.$$

Furthermore, if A and B are each alternating, then many terms in the summation become identical and we may remove the numerical factor $1/p!\,q!$. Since, in this paper, we shall confine our attention to the case of alternating

tensors, we shall write as our explicit expression for the $r_1r_2 \cdots r_p s_1 \cdots s_q$ component of the outer product of the alternating tensors A and B

$$(1) \qquad [AB]^{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!\,q!} \begin{bmatrix} r_1 \cdots r_p s_1 \cdots s_q \\ \alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

It is evident that this definition has no meaning if $p + q > n$, for in that case, the generalized Kronecker symbol vanishes identically.

With these definitions as a starting point, we shall proceed to prove a few theorems relating to the outer product of tensors.

We shall first prove that

$$[AB] = (-1)^{pr}[BA]$$

where A is a tensor of rank p, B is a tensor of rank r, and $p + r < n$.

By definition

$$[AB]^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 l_2 \cdots l_p\, l_{p+1} \cdots l_{p+r} \\ \alpha_1\alpha_2 \cdots \alpha_p\, \alpha_{p+1} \cdots \alpha_{p+r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\alpha_{p+1} \cdots \alpha_{p+r}}$$

$$[BA]^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 l_2 \cdots l_r\, l_{r+1} \cdots l_{r+p} \\ \alpha_1\alpha_2 \cdots \alpha_r\, \alpha_{r+1} \cdots \alpha_{r+p} \end{bmatrix} B^{\alpha_1 \cdots \alpha_r} A^{\alpha_{r+1} \cdots \alpha_{p+r}}.$$

In order to compare similar terms, let us rewrite $[BA]$ in a different manner.

$$[BA]^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad l_{p+r} \\ \alpha_{p+1} \cdots \alpha_{p+r}\, \alpha_1 \cdots \alpha_p \end{bmatrix} B^{\alpha_{p+1} \cdots \alpha_{p+r}} A^{\alpha_1 \cdots \alpha_p}.$$

In order to determine the sign, we must rearrange the α's in the symbol so as to make them agree with those in the sign factor of $[AB]$. To do this, α_1 must be moved over r positions to bring it to the first position, i. e. the sign will be $(-1)^r$; similarly it will take r interchanges to bring α_2 to the second position, etc. Therefore, the final sign, when all p of the α's have reached their proper positions will be $(-1)^{rp}$. Hence

$$(2) \qquad [AB] = (-1)^{pr}[BA].$$

From this it follows that if C is a tensor of rank s, and D is a tensor of rank t, then

$$\begin{aligned} [ABCD] &= (-1)^{st}\,[ABDC] = (-1)^{st+rt}\,[ADBC] \\ &= (-1)^{st+rt+pt}\,[DABC] = (-1)^{st+rt+pt+rs+ps+pr}\,[DCBA]. \end{aligned}$$

We shall now prove that

$$[b_1b_2 \cdot \cdot \cdot b_m] = \text{determinant of } \alpha\text{'s} \cdot [a_1a_2 \cdots a_m]$$

where $b_1, b_2, \cdots, b_m$ is a series of simple "extensive Grösse" which are linearly dependent upon m other simple "extensive Grösse" $a_1, a_2, \cdots, a_m$; i. e.

$$b_1 = \alpha_1{}^{\rho_1} a_{\rho_1}, \quad b_2 = \alpha_2{}^{\rho_2} a_{\rho_2} \cdots b_m = \alpha_m{}^{\rho_m} a_{\rho_m} .$$

Any component σ_1 of b_1 may be represented as $b_1{}^{\sigma_1} = \alpha_1{}^{\rho_1} a_{\rho_1}{}^{\sigma_1}$. By definition

$$[b_1 b_2 \cdots b_m]^{r_1 r_2 \cdots r_m} = \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} b_1{}^{\sigma_1} b_2{}^{\sigma_2} \cdots b_m{}^{\sigma_m}$$

$$= \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \alpha_1{}^{\rho_1} a_{\rho_1}{}^{\sigma_1} \cdot \alpha_2{}^{\rho_2} a_{\rho_2}{}^{\sigma_2} \cdots \alpha_m{}^{\rho_m} a_{\rho_m}{}^{\sigma_m}$$

$$= \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \alpha_1{}^{\rho_1} \alpha_2{}^{\rho_2} \cdots \alpha_m{}^{\rho_m} \cdot a_{\rho_1}{}^{\sigma_1} a_{\rho_2}{}^{\sigma_2} \cdots \cdots a_{\rho_m}{}^{\sigma_m}$$

$$= (1/m!) \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \begin{bmatrix} \lambda_1 \lambda_2 \cdots \lambda_m \\ 1 \; 2 \cdots m \end{bmatrix} \alpha_{\lambda_1}{}^{\rho_1} \alpha_{\lambda_2}{}^{\rho_2} \cdots \alpha_{\lambda_m}{}^{\rho_m} \cdot a_{\rho_1}{}^{\sigma_1} a_{\rho_2}{}^{\sigma_2} \cdots a_{\rho_m}{}^{\sigma_m} *$$

$$= (1/m!) \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \underset{1 \ldots m}{\overset{\rho_1 \ldots \rho_m}{D(\alpha)}} \quad a_{\rho_1}{}^{\sigma_1} a_{\rho_2}{}^{\sigma_2} \cdots \cdots a_{\rho_m}{}^{\sigma_m}$$

Hence

$$(3) \qquad [b_1 \cdots b_m]^{r_1 \cdots r_m} = \underset{1 \ldots m}{\overset{1 \ldots m}{D(\alpha)}} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} a_1{}^{\sigma_1} a_2{}^{\sigma_2} \cdots a_m{}^{\sigma_m}$$

$$= \text{Determinant of } \alpha\text{'s } [a_1 a_2 \cdots a_m].\dagger$$

So far, we have been concerned only with the outer product of tensors, the sum of whose ranks is less than or at most equal to n. But before proceeding to the case where this sum is greater than n, we shall define a new quantity known as the complement of a tensor.

If A is a tensor of rank p, then the complement of A is a tensor B of rank $n - p$ (represented as $|A \equiv B$). Any component of the complement of A is explicitly defined as

$$(4) \qquad (|A)_{s_1 \ldots s_{n-p}} = B_{s_1 \ldots s_{n-p}} = (1/p!) \begin{bmatrix} 1 \cdots \cdots n \\ s_1 \cdots s_{n-p} \, \rho_1 \cdots \rho_p \end{bmatrix} A^{\rho_1 \cdots \rho_p}.\ddagger$$

Before proceeding to the next proof, we shall introduce at this point several lemmas, concerning the generalized Kronecker symbol, as they will be needed in the following proofs.

---

* Formula 2.3 of Professor Murnaghan's paper, *American Mathematical Monthly.*

† Formula 3.4 of Professor Murnaghan's paper, *American Mathematical Monthly.*

‡ If A is a contravariant tensor of rank p, the complement of A is covariant of rank $n - p$. The shift of labels from above to below is used to indicate the change of type of tensor.

LEMMA I.

$$(5)\quad \begin{bmatrix} r_1 & \cdots & \cdots & r_m \\ s_1 \cdots s_g & \alpha_1 & \cdots & \alpha_{m-g} \end{bmatrix} \begin{bmatrix} \alpha_1 \cdots \alpha_{m-g} \\ p_1 \cdots p_{m-g} \end{bmatrix} = (m-g)! \begin{bmatrix} r_1 & \cdots & \cdots & r_m \\ s_1 \cdots s_g & p_1 & \cdots & p_{m-g} \end{bmatrix}.$$

From the second factor on the left-hand side, it is evident that in order that the symbol shall not be zero, $\alpha_1 \cdots \alpha_{m-g}$ must be some arrangement of the quantities $p_1, p_2, \cdots, p_{m-g}$. Any inversion in a particular set of values assigned to $\alpha_1, \alpha_2, \cdots, \alpha_{m-g}$ in the one factor will cause a similar inversion in the other factor; hence, their product will have the same sign as if the arrangements of $\alpha_1 \cdots \alpha_{m-g}$ and $p_1 \cdots p_{m-g}$ are identical. Since there are $(m-g)!$ such arrangements of $\alpha_1, \alpha_2, \cdots, \alpha_{m-g}$ we must introduce the factor $(m-g)!$ on the right.

LEMMA II.

$$(6)\quad \begin{bmatrix} r_1 & \cdots & \cdots & r_m \\ s_1 \cdots s_g & \alpha_1 & \cdots & \alpha_{m-g} \end{bmatrix} \begin{bmatrix} b_1 \cdots b_g & \alpha_1 & \cdots & \alpha_{m-g} \\ r_1 & \cdots & \cdots & r_m \end{bmatrix} = (m-g)! \begin{bmatrix} b_1 \cdots b_g \\ s_1 \cdots s_g \end{bmatrix}$$

It is evident that if the symbol is not to be zero, that $s_1, s_2, \cdots, s_g$ is contained in $r_1, \cdots, r_m$; also $b_1, \cdots, b_g$ is contained in $r_1, \cdots, r_m$; therefore $\alpha_1, \cdots, \alpha_{m-g}$ may assume only such values of $r_1, \cdots, r_m$ as are not contained in $s_1, \cdots, s_g$. Consequently, $b_1, \cdots, b_g$ must contain only those values of $r, \cdots, r_m$ not assumed by $\alpha_1, \cdots, \alpha_{m-g}$, i. e. those contained in $s_1, \cdots, s_g$. Since there are $(m-g)!$ such arrangements of $\alpha_1, \cdots, \alpha_{m-g}$ the truth of (6) follows.

LEMMA III.

$$(7)\quad \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} A^{\alpha_1 \cdots \alpha_m} = \frac{m!}{g!(m-g)!} \begin{bmatrix} r_1 \cdots r_g \\ \alpha_1 \cdots \alpha_g \end{bmatrix} \times \begin{bmatrix} r_{g+1} \cdots r_m \\ \alpha_{g+1} \cdots \alpha_m \end{bmatrix} A^{\alpha_1 \cdots \alpha_m}$$

$$\begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} A^{\alpha_1 \cdots \alpha_m} = m!\, A^{r_1 \cdots r_m}$$

$$\begin{bmatrix} r_1 \cdots r_g \\ \alpha_1 \cdots \alpha_g \end{bmatrix} \begin{bmatrix} r_{g+1} \cdots r_m \\ \alpha_{g+1} \cdots \alpha_m \end{bmatrix} A^{\alpha_1 \cdots \alpha_m} = g!(m-g)!\, A^{r_1 \cdots r_m}$$

Both of these statements follow from the alternating character of these tensors.

Therefore the truth of (7) follows.

LEMMA IV.

(8) $$\begin{bmatrix} r_1 \cdots r_m \, \lambda_1 \cdots \lambda_r \\ \alpha_1 \cdots \cdots \alpha_{m+r} \end{bmatrix} \begin{bmatrix} 1 \cdots \cdots n \\ \lambda_1 \cdots \lambda_r \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}$$

$$= \frac{(m+r)\,!}{m\,!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} 1 \cdots \cdots n \\ \alpha_{m+1} \cdots \alpha_{m+r} \, \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}.$$

From (7) we have

$$\begin{bmatrix} r_1 \cdots r_m \, \lambda_1 \cdots \lambda_r \\ \alpha_1 \cdots \alpha_m \, \alpha_{m+1} \cdots \alpha_{m+r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} = \frac{(m+r)\,!}{m\,!\,r\,!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda_1 \cdots \cdots \lambda_r \\ \alpha_{m+1} \cdots \alpha_{m+r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}}.$$

Substituting this in the left-hand member of (8) we obtain

$$\frac{(m+r)\,!}{m\,!\,r\,!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} \lambda_1 \cdots \cdots \lambda_r \\ \alpha_{m+1} \cdots \alpha_{m+r} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \cdots \cdots n \\ \lambda_1 \cdots \lambda_r \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}.$$

$$= \frac{(m+r)\,!}{m\,!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} 1 \cdots \cdots n \\ \alpha_{m+1} \cdots \alpha_{m+r} \, \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}.$$

(by 5)

It is readily shown that *the complement of the complement of a tensor is equal to* $\pm$ *the original tensor.*

If A is a tensor of rank p, then $B = |\, A$ is a tensor of rank $n - p$, and $C = |\, B$ is a tensor of rank p. We shall prove therefore that

$$||\, A = \pm A.$$

By definition,

$$C^{p_1 \cdots p_p} = \frac{1}{(n-p)\,!} \begin{bmatrix} p_1 \cdots p_p \, \alpha_1 \cdots \alpha_{n-p} \\ 1 \cdots \cdots n \end{bmatrix} [\,|\, B]_{\alpha_1 \cdots \alpha_{n-p}}$$

$$= \frac{1}{(n-p)\,!\,p\,!} \begin{bmatrix} p_1 \cdots p_p \, \alpha_1 \cdots \alpha_{n-p} \\ 1 \cdots \cdots n \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \cdots \cdots n \\ \alpha_1 \cdots \alpha_{n-p} \, \rho_1 \cdots \rho_p \end{bmatrix} A^{\rho_1 \cdots \rho_p} \quad \text{(by 4)}$$

$$= \frac{(-1)^{p(n-p)}}{p\,!} \begin{bmatrix} p_1 \cdots p_p \\ \rho_1 \cdots \rho_p \end{bmatrix} A^{\rho_1 \cdots \rho_p} \quad \text{(by 6)}$$

$$= (-1)^{p(n-p)} A^{p_1 \cdots p_p}$$

(9) $$\therefore \quad ||\, A = (-1)^{p(n-p)} A.$$

If A is of order p and B is of order r, and $p+r>n$, then as we noted above, our former definition of the outer product of two tensors (when $p+r<n$) would have no meaning. Hence, we define the outer product of AB, when $p+r>n$ as a tensor of rank $p+r-n$ and its explicit expression is found by taking the outer product of the complements of A and of B, and taking the complement of this outer product i. e. $|\,[|A\cdot|B]=[AB]$ by definition, when $p+r>n$. $|A$ is a tensor of rank $n-p$; $|B$ is a tensor of rank $n-r$; $[|A\cdot|B]$ is a tensor of rank $2n-p-r<n$; $|\,[\,|A\cdot|B]$ is a tensor of rank $p+r-n<n$.

To find the explicit expression for this outer product, we proceed directly.

$$[|A\cdot|B]_{l_1\ldots l_{2n-p-r}}=\frac{1}{p!\,r!(n-p)!\,(n-r)!}\begin{bmatrix}\mu_1\cdots\mu_{n-p}\,\tau_1\cdots\tau_{n-r}\\ l_1\;\cdots\;\cdots\;l_{2n-p-r}\end{bmatrix}$$

$$\times\begin{bmatrix}1\;\cdots\cdots\;n\\ \mu_1\cdots\mu_{n-p}\,\alpha_1\cdots\alpha_p\end{bmatrix}\begin{bmatrix}1\;\cdots\cdots\;n\\ \tau_1\cdots\tau_{n-r}\,\beta_1\cdots\beta_r\end{bmatrix}A^{\alpha_1\ldots\alpha_p}B^{\beta_1\ldots\beta_r}$$

Taking the complement of this, we obtain

$$|\,[\,|A\cdot|B]_{l_1\ldots l_{2n-p-r}}=C^{m_1\ldots m_{p+r-n}}$$

$$=\frac{1}{p!\,r!(n-p)!(n-r)!(2n-p-r)!}$$

$$\begin{bmatrix}m_1\cdots m_{p+r-n}\,\rho_1\cdots\rho_{2n-p-r}\\ 1\;\cdots\cdots\cdots\;n\end{bmatrix}\begin{bmatrix}\mu_1\cdots\mu_{n-p}\,\tau_1\cdots\tau_{n-r}\\ \rho_1\;\cdots\;\rho_{2n-p-r}\end{bmatrix}$$

$$\times\begin{bmatrix}1\;\cdots\cdots\;n\\ \mu_1\cdots\mu_{n-p}\,\alpha_1\cdots\alpha_p\end{bmatrix}\begin{bmatrix}1\;\cdots\cdots\;n\\ \tau_1\cdots\tau_{n-r}\,\beta_1\cdots\beta_r\end{bmatrix}A^{\alpha_1\ldots\alpha_p}B^{\beta_1\ldots\beta_r}$$

$$=\frac{(-1)^{(n-p)(p+r-n)}}{p!\,r!(n-r)!}\begin{bmatrix}m_1\cdots m_{p+r-n}\,\tau_1\cdots\tau_{n-r}\\ \alpha_1\;\cdots\cdots\cdots\;\alpha_p\end{bmatrix}$$

$$\times\begin{bmatrix}1\;\cdots\cdots\;n\\ \tau_1\cdots\tau_{n-r}\,\beta_1\cdots\beta_r\end{bmatrix}A^{\alpha_1\ldots\alpha_p}B^{\beta_1\ldots\beta_r}$$

(by 5 and 6).

$$(10)\qquad=\frac{(-1)^{(n-p)(r-1)}}{r!(n-r)!(p+r-n)!}\begin{bmatrix}m_1\cdots m_{p+r-n}\\ \alpha_1\cdots\alpha_{p+r-n}\end{bmatrix}$$

$$\times\begin{bmatrix}1\;\cdots\cdots\cdots\;n\\ \alpha_{p+r-n+1}\cdots\alpha_p\,\beta_1\cdots\beta_r\end{bmatrix}A^{\alpha_1\ldots\alpha_p}B^{\beta_1\ldots\beta_r}$$

By means of this expression, it is readily shown that *the complement of the outer product of two tensors is equal to* $\pm$ *the outer product of their complements,* i. e.,

$$|\,[AB]=\pm[|A\cdot|B]$$

where A is a tensor of rank p, B is a tensor of rank r.

Case I. $p + r < n$.

$|\,[AB]$ is of rank $n - p - r$.

$[|\,A \cdot |\,B]$ is of rank $2n - p - r - n = n - p - r < n$.

$$|\,[AB] = |\,C^{l_1 \cdots l_{p+r}}$$

$$C^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 & \cdots & \cdots & l_{p+r} \\ \alpha_1 \cdots \alpha_p & \beta_1 & \cdots & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

$$|\,C^{l_1 \cdots l_{p+r}} = D_{s_1 \cdots s_{n-p-r}}$$

$$= \frac{1}{p!\,r!\,(p+r)!} \begin{bmatrix} 1 & \cdots & \cdots & n \\ s_1 \cdots s_{n-p-r} & \lambda_1 & \cdots & \lambda_{p+r} \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda_1 & \cdots & \cdots & \lambda_{p+r} \\ \alpha_1 \cdots \alpha_p & \beta_1 & \cdots & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

$$= \frac{1}{p!\,r!} \begin{bmatrix} 1 & \cdots & \cdots & \cdots & n \\ s_1 \cdots s_{n-p-r} & \alpha_1 \cdots \alpha_p & \beta_1 & \cdots & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

(by 5).

To find $[|\,A \cdot |\,B]$ we must use the expression for the outer product of two tensors, the sum of whose ranks is greater than n.

$$[|\,A \cdot |\,B]_{s_1 \cdots s_{n-p-r}} = \frac{(-1)^{p(n-p-r)}}{(n-p-r)!\,(n-r)!\,r!} \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p-r} \\ s_1 \cdots s_{n-p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_{n-p-r+1} \cdots \alpha_{n-p} & \beta_1 \cdots \beta_{n-r} \\ 1 \quad \cdots \quad \cdots & \cdots \quad n \end{bmatrix} (|\,A)_{\alpha_1 \cdots \alpha_{n-p}} (|\,B)_{\beta_1 \cdots \beta_{n-r}}$$

$$= \frac{(-1)^{p(n-p-r)}}{(n-p-r)!\,(n-r)!\,r!^2 p!} \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p-r} \\ s_1 \cdots s_{n-p-r} \end{bmatrix} \begin{bmatrix} \alpha_{n-p-r+1} \cdots \alpha_{n-p} & \beta_1 \cdots \beta_{n-r} \\ 1 \quad \cdots \quad \cdots & \cdots \quad n \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdots & \cdots & n \\ \alpha_1 \cdots \alpha_{n-p} & \gamma_1 & \cdots & \gamma_p \end{bmatrix} \begin{bmatrix} 1 & \cdots & \cdots & n \\ \beta_1 \cdots \beta_{n-r} & \delta_1 & \cdots & \delta_r \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} B^{\delta_1 \cdots \delta_r}$$

$$= \frac{(-1)^{(p+r)(n-1)}}{r!\,p!} \begin{bmatrix} 1 & \cdots & \cdots & \cdots & n \\ s_1 \cdots s_{n-p-r} & \gamma_1 \cdots \gamma_p & \delta_1 & \cdots & \delta_r \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} B^{\delta_1 \cdots \delta_r}$$

(by 5, 6, 7)

$$\therefore\ |\,[AB] = (-1)^{(p+r)(n-1)} [|\,A \cdot |\,B].$$

Case II. $p + r > n$.

$[|\,A \cdot |\,B]$ is a tensor of rank $2n - p - r < n$.

$|\,[AB]$ is a tensor of rank $2n - p - r$.

$| [AB] = | [| A \cdot | B]$ by definition.

$\therefore \quad | [AB] = || [| A \cdot | B]. \quad = (-1)^{(2n-p-r)(n-1)} [| A \cdot | B]$ (by 9)

(11) $$= (-1)^{(p+r)(n-1)} [| A \cdot | B].$$

Hence the theorem holds when $p + r < n$ and $p + r > n$.

We shall next prove $[A(BC)] = [ABC]$ where A, B, C are tensors of rank p, r, s respectively.

Case I. $p + r + s < n$.

$$[A(BC)]^{l_1 \cdots l_{p+r+s}} = \frac{1}{(r+s)!\,p!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & l_{p+r+s} \\ \alpha_1 & \cdots & \alpha_p & \beta_1 & \cdots & \beta_{r+s} \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} [BC]^{\beta_1 \cdots \beta_{r+s}}$$

$$= \frac{1}{(r+s)!\,p!\,r!\,s!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & l_{p+r+s} \\ \alpha_1 & \cdots & \alpha_p & \beta_1 & \cdots & \beta_{r+s} \end{bmatrix}$$

$$\times \begin{bmatrix} \beta_1 & \cdot & \cdot & \cdot & \beta_{r+s} \\ \gamma_1 & \cdots & \gamma_r & \delta_1 & \cdots & \delta_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\gamma_1 \cdots \gamma_r} C^{\delta_1 \cdots \delta_s}$$

$$= \frac{1}{p!\,r!\,s!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & l_{p+r+s} \\ \alpha_1 & \cdots & \alpha_p & \gamma_1 & \cdots & \gamma_r & \delta_1 & \cdots & \delta_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\gamma_1 \cdots \gamma_r} C^{\delta_1 \cdots \delta_s}$$

$$= [ABC]^{l_1 \cdots l_{p+r+s}}.$$

Hence

(12) $$[A(BC)] = [ABC].$$

Case II. $p + r + s > 2n$.

Let $A' = | A$, $B' = | B$, $C' = | C$. Then A' is a tensor of rank $n - p$; B' is a tensor of rank $n - r$; C' is a tensor of rank $n - s$. $[A'B'C']$ is a tensor of rank $3n - p - r - s < n$ and therefore comes under Case I.

$\therefore \quad [A'(B'C')]^{l_1 \cdots l_{3n-p-r-s}} = [A'B'C']^{l_1 \cdots l_{3n-p-r-s}}$

$| [A'B'C'] = (-1)^{(3n-p-r-s)(n-1)} [| A' \cdot | B' \cdot | C']$. But $A' = | A$ and

$| A' = || A = (-1)^{p(n-1)} A$.

$\therefore \quad | [A'B'C'] = (-1)^{(3n-p-r-s+p+r+s)(n-1)} [ABC] = (-1)^{3n(n-1)} [ABC]$
$= [ABC]$.

Similarly $| [A'(B'C')] = (-1)^{(3n-p-r-s)(n-1)} [| A' \cdot | (B'C')]$
$= (-1)^{3n(n-1)} [A(BC)] = [A(BC)]$.

Since $[A'(B'C')] = [A'B'C']$,

then $[A(BC)] = [ABC]$.

The explicit expression for $[ABC]$ when $p+r+s>2n$ may be derived as follows:

$$| \, [A'B'C'] = [ABC]^{s_1 \cdots s_{p+r+s-2n}}$$

$$= \frac{1}{(3n-p-r-s)\,!\,(n-r)\,!\,(n-s)\,!\,(n-p)\,!\,r\,!\,p\,!\,s\,!} \begin{bmatrix} s_1 \cdots s_{p+r+s-2n}\, \lambda_1 \cdots \lambda_{3n-p-r-s} \\ 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n \end{bmatrix}$$

$$\times \begin{bmatrix} \epsilon_1 \cdots \epsilon_{n-p}\, \delta_1 \cdots \delta_{n-r}\, \sigma_1 \cdots \sigma_{n-s} \\ \lambda_1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, \lambda_{3n-p-r-s} \end{bmatrix} \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \, n \\ \epsilon_1 \cdots \epsilon_{n-p}\, \alpha_1 \cdots \alpha_p \end{bmatrix} \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \, n \\ \delta_1 \cdots \delta_{n-r}\, \beta_1 \cdots \beta_r \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \cdot \, n \\ \sigma_1 \cdots \sigma_{n-s}\, \gamma_1 \cdots \gamma_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r} C^{\gamma_1 \cdots \gamma_s}$$

$$= \frac{(-1)^{(p+r+s-2n)(n-p)}}{(p+r+s-2n)\,!\,(n-r)\,!\,(n-s)\,!\,r\,!\,s\,!} \begin{bmatrix} s_1 \cdots s_{p+r+s-2n} \\ \alpha_1 \cdots \alpha_{p+r+s-2n} \end{bmatrix}$$

$$\text{(12A)} \qquad \times \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n \\ \alpha_{p+r+s-2n+1} \cdots \alpha_{p+s-n}\, \beta_1 \cdots \beta_r \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n \\ \alpha_{p+s-n+1} \cdots \alpha_p\, \gamma_1 \cdots \gamma_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r} C^{\gamma_1 \cdots \gamma_s}$$

We shall further prove that

$$\text{(13)} \qquad [AB \cdot AC] = \pm \, [ABC]A.$$

where A, B, C are tensors of rank p, r, s respectively.

Case I. $p+r+s=n$.

$[\,|\, AB \cdot |\, AC]$ is a tensor of rank $s+r$ or $n-p$,

$|\, [\,|\, AB \cdot |\, AC]$ is a tensor of rank p.

$$[AB \cdot AC]^{m_1 \cdots m_p} = \frac{(-1)^{ps}}{p\,!\,(p+s)\,!\,r\,!} \begin{bmatrix} m_1 \cdots m_p \\ \alpha_1 \cdots \alpha_p \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n \\ \alpha_{p+1} \cdots \alpha_{p+r}\, \beta_1 \cdots \beta_{p+s} \end{bmatrix} [AB]^{\alpha_1 \cdots \alpha_{p+r}} [AB]^{\beta_1 \cdots \beta_{p+s}}$$

$$= \frac{(-1)^{ps}}{p\,!^3\,(p+s)\,!\,r\,!^2\,s\,!} \begin{bmatrix} m_1 \cdots m_p \\ \alpha_1 \cdots \alpha_p \end{bmatrix} \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \cdot \, n \\ \alpha_{p+1} \cdots \alpha_{p+r}\, \beta_1 \cdots \beta_{p+s} \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_1 \cdot \cdot \cdot \, \alpha_{p+r} \\ \gamma_1 \cdots \gamma_p\, \delta_1 \cdots \delta_r \end{bmatrix} \begin{bmatrix} \beta_1 \cdot \cdot \cdot \, \beta_{p+s} \\ \mu_1 \cdots \mu_p\, \rho_1 \cdots \rho_s \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} B^{\delta_1 \cdots \delta_r} A^{\mu_1 \cdots \mu_p} C^{\rho_1 \cdots \rho_s}$$

$$= \frac{(-1)^{p(s+r)}}{p\,!^2\,r\,!\,s\,!} \begin{bmatrix} m_1 \cdots m_p \\ \gamma_1 \cdots \gamma_p \end{bmatrix} \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n \\ \mu_1 \cdots \mu_p\, \delta_1 \cdots \delta_r\, \rho_1 \cdots \rho_s \end{bmatrix}$$

$$\times \, A^{\gamma_1 \cdots \gamma_p} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\rho_1 \cdots \rho_s}$$

$$= \frac{(-1)^{p(s+r)}}{p!}\begin{bmatrix} m_1 \cdots m_p \\ \gamma_1 \cdots \gamma_p \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} \cdot$$

$$\times \frac{1}{p!\,r!\,s!}\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \mu_1 \cdots \mu_p & \delta_1 \cdots \delta_r & \rho_1 \cdots \rho_s \end{bmatrix} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\rho_1 \cdots \rho_s}$$

(13A) $$= (-1)^{p(s+r)}[ABC]\, A^{m_1 \cdots m_p}.$$

Case II. $p + r + s = 2n.$

Let $A' = |\,A,\ \ B' = |\,B,\ \ C' = |\,C$. Then $[A'B'C']$ is a tensor of rank $3n - p - r - s = n$. Hence we can apply the theorem for Case I.

$$\therefore \quad [A'B' \cdot A'C'] = (-1)^{(n-p)(n-1)}\ [A'B'C']A'$$

$$|\,[A'B' \cdot A'C'] = (-1)^{(2n-p)(n-1)}\ [|\,A' \cdot |\,B' \cdot |\,A' \cdot |\,C']$$
$$= (-1)^{(2n+p+r+s)(n-1)}[AB \cdot AC]$$
$$= [AB \cdot AC].$$

Therefore to find the expression for $[AB \cdot AC]$ when $p + r + s = 2n$ we shall find the complement of $[A'B' \cdot A'C']$

$$[A'B' \cdot A'C']_{l_1 \ldots l_{n-p}} = \frac{(-1)^{p(n-1)}}{(n-p)\,!^2(n-r)\,!(n-s)\,!}\begin{bmatrix} \alpha_1 \cdots \alpha_{n-p} \\ l_1 \cdots l_{n-p} \end{bmatrix}$$

$$\times \begin{bmatrix} \rho_1 \cdots \rho_{n-p}\ \sigma_1 \cdots \sigma_{n-r}\ \epsilon_1 \cdots \epsilon_{n-s} \\ 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot n \end{bmatrix}$$

$$\times A'_{\alpha_1 \ldots \alpha_{n-p}}\ A'_{\rho_1 \ldots \rho_{n-p}}\ B'_{\sigma_1 \ldots \sigma_{n-r}}\ C'_{\epsilon_1 \ldots \epsilon_{n-s}} \qquad \text{(Case I)}$$

$$= \frac{(-1)^{p(n-1)}}{p\,!^2 r\,!s\,!(n-p)\,!^2(n-r)\,!(n-s)\,!}\begin{bmatrix} \alpha_1 \cdots \alpha_{n-p} \\ l_1 \cdots l_{n-p} \end{bmatrix}$$

$$\times \begin{bmatrix} \rho_1 \cdots \rho_{n-p}\ \sigma_1 \cdots \sigma_{n-r}\ \epsilon_1 \cdots \epsilon_{n-s} \\ 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot n \end{bmatrix}\begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot \cdot n \\ \alpha_1 \cdots \alpha_{n-p}\ \gamma_1 \cdots \gamma_p \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot n \\ \rho_1 \cdots \rho_{n-p}\ \mu_1 \cdots \mu_p \end{bmatrix}\begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot n \\ \sigma_1 \cdots \sigma_{n-r}\ \delta_1 \cdots \delta_r \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \cdot \cdot \cdot \cdot \cdot n \\ \epsilon_1 \cdots \epsilon_{n-s}\ \nu_1 \cdots \nu_s \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

Taking the complement of this, we obtain

$$|\,[A'B' \cdot A'C'] = [AB \cdot AC]^{m_1 \cdots m_p}$$

$$= \frac{(-1)^{p(n-1)}}{p\,!^2 r\,!s\,!(n-p)\,!^3(n-r)\,!(n-s)\,!}\begin{bmatrix} m_1 \cdots m_p\ \lambda_1 \cdots \lambda_{n-p} \\ 1 \cdot \cdot \cdot \cdot \cdot \cdot n \end{bmatrix}$$

$$\times \begin{bmatrix}\alpha_1 \cdots \alpha_{n-p}\\ \lambda_1 \cdots \lambda_{n-p}\end{bmatrix}\begin{bmatrix}\rho_1 \cdots \rho_{n-p}\,\sigma_1 \cdots \sigma_{n-r}\,\epsilon_1 \cdots \epsilon_{n-s}\\ 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n\end{bmatrix}$$

$$\times \begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \alpha_1 \cdots \alpha_{n-p}\,\gamma_1 \cdots \gamma_p\end{bmatrix}\begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \rho_1 \cdots \rho_{n-p}\,\mu_1 \cdots \mu_p\end{bmatrix}\begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \sigma_1 \cdots \sigma_{n-r}\,\delta_1 \cdots \delta_r\end{bmatrix}$$

$$\times \begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \epsilon_1 \cdots \epsilon_{n-s}\,\nu_1 \cdots \nu_s\end{bmatrix} A^{\gamma_1 \cdots \gamma_p} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

$$= \frac{1}{p!}\begin{bmatrix}m_1 \cdots m_p\\ \gamma_1 \cdots \gamma_p\end{bmatrix} A^{\gamma_1 \cdots \gamma_p} \cdot \frac{1}{r!s!(n-r)!(n-s)!}$$

$$\times \begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \mu_1 \cdots \mu_{n-r}\,\delta_1 \cdots \delta_r\end{bmatrix}\begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \cdot \, n\\ \mu_{n-r+1} \cdots \mu_p\,\nu_1 \cdots \nu_s\end{bmatrix}$$

$$\times A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

$$= A^{m_1 \cdots m_p}[ABC]. \tag{13B}$$

In order to show that the above expression is equal to $[ABC]$ when $p + r + s = 2n$, we shall first find $[A'B'C']$ whose rank $= n$.

$$[A'B'C'] = \frac{1}{(n-r)!(n-p)!(n-s)!}\begin{bmatrix}\alpha_1 \cdots \alpha_{n-p}\,\beta_1 \cdots \beta_{n-r}\,\rho_1 \cdots \rho_{n-s}\\ 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n\end{bmatrix}$$

$$\times \; A'_{\alpha_1 \ldots \alpha_{n-p}} \; B'_{\beta_1 \ldots \beta_{n-r}} \; C'_{\rho_1 \ldots \rho_{n-s}}.$$

But the complement of $[A'B'C']$ is $[ABC]$; i. e.

$$|\,[A'B'C'] = (-1)^{n(n-1)}[\,|\,A' \cdot |\,B' \cdot |\,C'] = [ABC]$$

$$\therefore \quad |\,[A'B'C'] = \frac{1}{n!(n-r)!(n-p)!(n-s)!p!r!s!}\begin{bmatrix}\sigma_1 \cdots \sigma_n\\ 1 \cdots n\end{bmatrix}$$

$$\times \begin{bmatrix}\alpha_1 \cdots \alpha_{n-p}\,\beta_1 \cdots \beta_{n-r}\,\rho_1 \cdots \rho_{n-s}\\ 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \, n\end{bmatrix}\begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \alpha_1 \cdots \alpha_{n-p}\,\mu_1 \cdots \mu_p\end{bmatrix}$$

$$\times \begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \beta_1 \cdots \beta_{n-r}\,\delta_1 \cdots \delta_r\end{bmatrix}\begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \, n\\ \rho_1 \cdots \rho_{n-s}\,\nu_1 \cdots \nu_s\end{bmatrix} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

$$= \frac{1}{(n-r)!(n-s)!r!s!}\begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \cdot \, n\\ \mu_1 \cdots \mu_{n-r}\,\delta_1 \cdots \delta_r\end{bmatrix} \tag{13C}$$

$$\times \begin{bmatrix}1 \cdot \cdot \cdot \cdot \cdot \cdot \, n\\ \mu_{n-r+1} \cdots \mu_p\,\nu_1 \cdots \nu_s\end{bmatrix} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

3. *Inner Product of Tensors.* In the study of simple vectors, we consider two varieties of products,—outer or vector products and inner or scalar products. So here, we form not only outer products of tensors, but also inner products.

The inner product of two "extensive Grösse" A, B, as defined by Grass-

mann, is the outer product of A into the complement of B; i. e. the inner product of A into B equals $[A \mid B]$. We note here that if A is of rank p and B is of rank r, and $p > r$, then the rank of $[A \mid B]$ is equal to $p - r$; but if $p < r$, then the rank of $[A \mid B]$ is $n + p - r$. In order to keep the two cases more nearly similar, we shall modify the above definition as follows.

If $p \geqq r$, the inner product of A into B is defined as $[A \mid B]$.
If $p < r$, the inner product of A into B is defined as $\mid [A \mid B]$.

The rank in the second case is now $r - p$.

We shall now proceed to derive the explicit expressions for the inner product of two tensors A and B, in the following cases:

I. The ranks of the two tensors are equal.
II. The rank of A is greater than rank of B.
III. The rank of A is less than rank of B.

Case I. A is a tensor of rank p, B is a tensor of rank r and $p = r$. Then the rank of $[A \cdot \mid B]$ is zero.

$$[A \mid B]^{s_1 \cdots s_n} = \frac{1}{p!p!(n-p)!}\begin{bmatrix} s_1 & \cdot & \cdot & \cdot & \cdot & s_n \\ \alpha_1 \cdots \alpha_p & & & & & \lambda_1 \cdots \lambda_{n-p} \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda_1 \cdots \lambda_{n-p} & \beta_1 \cdots \beta_p \\ 1 \quad \cdot \quad \cdot \quad \cdot & n \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_p}$$

$$= \frac{(-1)^{p(n-p)}}{p!\,p!}\begin{bmatrix} \beta_1 \cdots \beta_p \\ \alpha_1 \cdots \alpha_p \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_p}$$

$$= \frac{(-1)^{p(n-1)}}{p!} A^{\alpha_1 \cdots \alpha_p} B_{\alpha_1 \cdots \alpha_p} \tag{14}$$

Similarly

$$[B \mid A]_{s_1 \cdots s_n} = \frac{1}{p!\,p!(n-p)!}\begin{bmatrix} \beta_1 \cdots \beta_p & \mu_1 \cdots \mu_{n-p} \\ s_1 \quad \cdot \quad \cdot \quad \cdot & s_n \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot & n \\ \mu_1 \cdots \mu_{n-p} & \alpha_1 \cdots \alpha_p \end{bmatrix} B_{\beta_1 \cdots \beta_p} A^{\alpha_1 \cdots \alpha_p}$$

$$= \frac{(-1)^{p(n-1)}}{p!} A^{\alpha_1 \cdots \alpha_p} B_{\alpha_1 \cdots \alpha_p}.$$

Hence $[A \mid B] = [B \mid A]$ when $p = r$.

It is evident from (14) that Grassmann's definition of the inner product of two tensors is in accord with the usual method of finding the inner product of two tensors, that of multiplying scalarly the contravariant presentation of one into the covariant presentation of the other.

Case II. $p > r$.

By definition, the inner product of A into B is $[A \mid B]$. It is evident that the sum of the ranks of A and of $\mid B$ is $p + n - r$ which is greater than n, hence we must use formula 10 in writing the explicit expression for $[A \mid B]$. Thus

$$[A \mid B]^{m_1 \cdots m_{p-r}} = \frac{(-1)^{(p-r)(n-p)}}{(p-r)!(n-r)!\,r!} \begin{bmatrix} m_1 \cdots m_{p-r} \\ \alpha_1 \cdots \alpha_{p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \alpha_{p-r+1} & \cdots & \alpha_p & \beta_1 & \cdots & & \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} [\mid B]^{\beta_1 \cdots \beta_{n-r}}$$

$$(15) \qquad = \frac{(-1)^{(p-r)(n-p)+r(n-1)}}{(p-r)!\,r!^2} \begin{bmatrix} m_1 \cdots m_{p-r} \\ \alpha_1 \cdots \alpha_{p-r} \end{bmatrix} \begin{bmatrix} \gamma_1 & \cdots & \gamma_r \\ \alpha_{p-r+1} & \cdots & \alpha_p \end{bmatrix}$$

$$\times\ A^{\alpha_1 \cdots \alpha_p} B_{\gamma_1 \cdots \gamma_r}.$$

Case III. $p < r$.

By definition, the inner product of A into B in this case is $\mid [A \cdot \mid B]$. The rank of this tensor is $r - p$.

$$[A \mid B]^{s_1 \cdots s_{n+p-r}} = \frac{1}{p!(n-r)!\,r!} \begin{bmatrix} s_1 & \cdot & \cdot & \cdot & & s_{n+p-r} \\ \alpha_1 \cdots \alpha_p & \mu_1 & \cdots & & & \mu_{n-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \mu_1 \cdots \mu_{n-r} & \beta_1 \cdots \beta_r \\ 1 \quad \cdot \quad \cdot \quad \cdot & \cdot \quad n \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_r}.$$

Taking the complement of this, we obtain

$$\mid [A \mid B] = C_{l_1 \cdots l_{r-p}}$$

$$= \frac{1}{(n+p-r)!\,p!\,r!(n-r)!} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ l_1 \cdots l_{r-p} & \sigma_1 & \cdots & & & & \sigma_{n+p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \sigma_1 & \cdot & \cdot & \cdot & \sigma_{n+p-r} \\ \alpha_1 \cdots \alpha_p & \mu_1 & \cdots & & \mu_{n-r} \end{bmatrix} \begin{bmatrix} \mu_1 \cdots \mu_{n-r} & \beta_1 \cdots \beta_r \\ 1 \quad \cdot \quad \cdot \quad \cdot & \cdot \quad n \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_r}.$$

$$(16) \qquad = \frac{(-1)^{r(n-1)}}{(r-p)!\,p!^2} \begin{bmatrix} \beta_1 \cdots \beta_{r-p} \\ l_1 \cdots l_{r-p} \end{bmatrix} \begin{bmatrix} \beta_{r-p+1} \cdots \beta_r \\ \alpha_1 \quad \cdots \quad \alpha_p \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_r}.$$

Let us now compare the inner product of A into B with that of B into A.

I. When $p > r$.

$$[A \mid B]^{m_1 \cdots m_{p-r}} = \frac{(-1)^{(p-r)(n-p)+r(n-1)}}{(p-r)!\,r!^2} \begin{bmatrix} m_1 \cdots m_{p-r} \\ \alpha_1 \cdots \alpha_{p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \beta_1 & \cdots & \beta_r \\ \alpha_{p-r+1} & \cdots & \alpha_p \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_r}.$$

Now $[B \mid A]$ will fall under Case III, therefore by 16, we may write at once

$$[B \mid A]_{m_1 \cdots m_{p-r}} = \frac{(-1)^{p(n-1)}}{(p-r)!\, r!^2} \begin{bmatrix} \alpha_1 \cdots \alpha_{p-r} \\ m_1 \cdots m_{p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_{p-r+1} \cdots \alpha_p \\ \beta_1 \cdots \beta_r \end{bmatrix} A_{\alpha_1 \cdots \alpha_p} \; B^{\beta_1 \cdots \beta_r}.$$

The two expressions are of different types, but otherwise, they are identical except for sign. However, if we multiply $[B \mid A]$ by $(-1)^{r(p-1)}$, the two are equal, since the exponents of (-1) are congruent, modulus 2.

II. When $p < r$, the same relation holds, since in either case, it results in a comparison of formulas 15 and 16. Hence the inner product of A into $B = (-1)^{r(p-1)}$ inner product of B into A, provided we do not distinguish between contravariant and covariant.

4. *Conclusion.* In the preceding pages, we have endeavored to give a modern presentation of Grassmann's *Die Ausdehnungslehre* with special emphasis on the development of the explicit expressions for various combinations of tensors.

Having defined, by means of the generalized Kronecker symbol, the outer product of tensors, the sum of whose rank $< n$, and also the complement of a tensor, we have derived the explicit expression for the outer product of tensors, the sum of whose ranks $> n$,

From these expressions as a basis, various relations between the outer products of tensors are readily derived.

We have also derived expressions for the inner product of two tensors, of ranks p and q, in the cases where $p = r$, $p > r$ and $p < r$.*

* Since writing the above paper, we have read with interest a paper by C. L. E. Moore, "Grassmannian Geometry in Riemannian Space," *Journal of Mathematics and Physics* (Massachusetts Institute of Technology), Vol. V, No. 4 (June, 1926), which bears directly on this subject.

e

l
e

s
o
h

a
-
s

r
f
-

r

s,

t.
d
h

THE JOHNS HOPKINS PRESS
SERIAL PUBLICATIONS

American Journal of Mathematics. Edited by G. D. Birkhoff, A. B. Coble, Abraham Cohen, G. C. Evans and Frank Morley (Chairman). Quarterly. 8vo. Volume XLIX in progress. $6 per volume. (Foreign postage, twenty-five cents.)

American Journal of Philology. Edited by C. W. E. Miller, with the coöperation of M. Bloomfield, H. Collitz, T. Frank, W. P. Mustard, and D. M. Robinson. Quarterly. 8vo. Volume LXVIII in progress. $5 per volume. (Foreign postage, twenty-five cents.)

American Journal of Psychiatry. E. N. Brush, C. M. Campbell, A. M. Barrett, G. H. Kirby, and H. Douglas Singer, Editors. Quarterly. 8vo. Volume VII in progress. $5 per volume. (Foreign postage, fifty cents.)

Biologia Generalis. (International Journal of General Biology). Edited by Leopold Löhner, Graz; Raymond Pearl, Baltimore, and Vladislav Růžička, Prague. 8vo. Volume III in progress. $20 per volume.

Bulletin of the American Association of Collegiate Registrars. Quarterly. 8vo. Series III in progress. $3 per year.

Hesperia. Hermann Collitz and Kemp Malone, Editors. 8vo. Twenty-four numbers have appeared.

Johns Hopkins Hospital Reports. 8vo. Volume XXII in progress. $5 per volume (Foreign postage, fifty cents.)

Johns Hopkins University Circular, including the President's Report, Annual Register, and Catalogue of the School of Medicine. Ten times yearly. 8vo. $1 per year.

Johns Hopkins University Studies in Archaeology. David M. Robinson, Editor. 8vo. Number two has appeared

Johns Hopkins University Studies in Education. Edward F. Buchner, Editor. 8vo. Nine numbers have appeared.

Johns Hopkins University Studies in Geology. Edward B. Mathews, Editor. 8vo. Seven numbers have been published.

Johns Hopkins University Studies in Historical and Political Science. Under the direction of the Departments of History, Political Economy and Political Science. 8vo. Volume XLV in progress. $5 per volume.

Johns Hopkins Studies in Romance Literatures and Languages. D. S. Blondheim, Gilbert Chinard, and H. C. Lancaster, Editors. 8vo. Twelve numbers have been published.

Modern Language Notes. Edited by Edwin Greenlaw, G. Gruenbaum, W. Kurrelmeyer, and H. C. Lancaster. Eight times yearly. 8vo. Volume XLII in progress. $5 per volume. (Foreign postage, fifty cents.)

Reprint of Economic Tracts. J. H. Hollander, Editor. Three series have appeared.

Terrestrial Magnetism and Atmospheric Electricity. L. A. Bauer, Editor. Quarterly. 8vo. Vol. XXXII in progress. $3.50 per volume.

Subscriptions and remittances should be sent to The Johns Hopkins Press, Baltimore, Md., U. S. A.